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Research Article

Lagrangian Relaxation for an Inventory Location Problem with Periodic Inventory Control and Stochastic Capacity Constraints

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We studied a joint inventory location problem assuming a periodic review for inventory control. A single plant supplies a set of products to multiple warehouses and they serve a set of customers or retailers. The problem consists in determining which potential warehouses should be opened and which retailers should be served by the selected warehouses as well as their reorder points and order sizes while minimizing the total costs. The problem is a Mixed Integer Nonlinear Programming (MINLP) model, which is nonconvex in terms of stochastic capacity constraints and the objective function. We propose a solution approach based on a Lagrangian relaxation and the subgradient method. The decomposition approach considers the relaxation of different sets of constraints, including customer assignment, warehouse demand, and variance constraints. In addition, we develop a Lagrangian heuristic to determine a feasible solution at each iteration of the subgradient method. The proposed Lagrangian relaxation algorithm provides low duality gaps and near-optimal solutions with competitive computational times. It also shows significant impacts of the selected inventory control policy into total system costs and network configuration, when it is compared with different review period values.

1. Introduction

Aggressive competition and strong economic turbulence in today's global markets drive companies to improve the performance of their supply chains in order to achieve a sustainable competitive advantage. The performance of a supply chain depends strongly on its design. Hence the managers' focus is there. In this context, supply chain network design (SCND) is a widely studied problem, which currently plays an important role in supply chain management and logistics [1, 2]. SCND consists of locating plants, warehouses, and distribution centers, allocating customers to open facilities while minimizing system-wide costs and satisfying service level requirements. Historically, the SCND problem has been tackled through a sequential approach that omits related tactical and operational decisions (e.g., inventory control, fleet design, and warehouse design). In this way, the omitted decisions are addressed after SCND has been solved. This means that strategic decisions, like the facility location, are made without regard to tactical decisions such as inventory control policy. This implies obtaining suboptimal SCND configurations because tactical decisions are subordinates to this network design [3].

This paper is focused on a three-level supply chain, where a single plant serves a set of warehouses, as Figure 1 shows. This set of warehouses serves a set of end retailers in a single commodity scenario. Unlike major previous inventory location models that assume a continuous review policy for warehouse inventory control, we use a periodic review policy (R, s, S) for each warehouse, where R is the period review, s is the reorder point, and S is the inventory objective level.

Thus, we study an inventory location model, in which stochastic inventory capacity constraints, expected inventory, and ordering costs are defined using a periodic review strategy. We formulate this inventory location model with periodic review control using an analysis of the expected

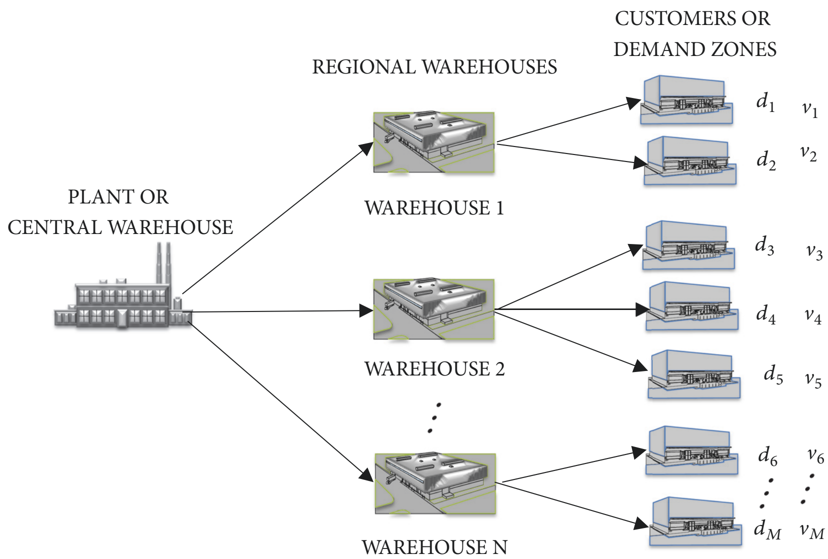

Figure 1: Representation of a distribution network of three stages.

safety stock, cyclic inventory and order quantities, and peak inventory levels for each potential warehouse. This MINLP model is NP-hard because it is an extension of the Capacitated Facility Location Problem (CFLP), which is already NP-hard.

Considering the high complexity of the analyzed problem, we propose an approximate solution approach based on Lagrangian relaxation and the subgradient method. The decomposition approach considers the relaxation of a different combination of problem constraints, including customer assignment, warehouse demand, and variance constraints. Then, we decompose the relaxed problem in a subproblem for each warehouse, which in turn is disaggregated in an inventory and location subproblem. In addition, a Lagrangian heuristic is developed to achieve a feasible solution at each iteration of the subgradient method. This Lagrangian heuristic is made up of warehouse selection and retailers greedy assignment, followed by local search improvements. We solve instances up to 20 potential warehouses and 40 retailers. The Lagrangian relaxation algorithm proposed in this paper provides low duality gaps and near-optimal solutions with competitive computational times. These results imply that this solution approach may be used in larger problem instances and more complex inventory location problems (ILP) as multicommodity and multiperiod formulations. In addition, the inclusion of periodic review policy in this model is relevant for those companies in which a continuous review policy is not feasible or there is a need to reduce costs for the inventory control system, especially for items in high demand. Considering all these attributes, ILP models could represent more accurately the complexity faced by distribution companies today.

This paper is organized as follows. In Section 2, we review the literature related to inventory location models. In Section 3, we discuss inventory control and capacity constraint issues. In Section 4, we present the formulation of the inventory location model with periodic review and stochastic capacity constraints. Section 5 presents the proposed solution approach based on Lagrangian relaxation. Section 6 presents and analyzes the numerical results. Finally, Section 7 presents conclusions, managerial insights, and suggestions for future research.

2. Literature Review

Over the last twenty years, several authors have studied how the inventory control decisions impact the Facility Location Problem (FLP) through the different integrated inventory location models. Barahona and Jensen [4] present an integer programming (IP) model for the location of a plant with cycle inventory costs, that is, the inventory required to satisfy the demands between two consecutive orders. These inventory costs are incorporated into the objective function as parameters, constituting a third term that is added to the fixed facility costs and transportation costs of Uncapacitated Fixed Charge Location Problem (UFLP). The linear relaxation of the model is solved through Dantzig-Wolfe decomposition. Nozick and Turnquist [5] develop a linear approach to the safety stock of a set of products based on the number of distribution centers through a simple linear regression. This allows safety stock costs to be directly included in the fixed cost coefficient of the UFLP. The resolution of the model is carried out through a hybrid heuristic established by Daskin [6]. Using the same previous framework, Nozick and Turnquist [7] expand their analysis by now considering a two-tier system (plant or central warehouse and DCs), where decisions are made considering whether products should have safety stock on the DCs or at the plant. Nozick and Turnquist [8] modify the previous formulations [5, 7] and now present a maximum covering location model, which ensures finding a proportion of the demand that meets a specific "coverage" distance of a DC. Later, using the approach proposed by Nozick and Turnquist [5], Lin et al. [9] solve a strategic design model of a multilevel and multiproduct distribution system, incorporating economies of scale in transportation and safety stock levels of the various products that are kept on the DCs through a greedy heuristic. All the previous models presented incorporate the operation stock and safety stock costs indirectly in the objective function and therefore, a linear term is added to it, so these models are classified as mixed integer programming models (MIP).

Erlebacher and Meller [10] are the first researchers to formulate a MINLP to address the ILP, in which the locations of the clients are continuously represented. Later Daskin et al. [11] present a location model of DCs that incorporate working and safety inventory costs, extending the UFLP model. In addition, the model includes transport costs from suppliers to DCs that explicitly combine economies of scale into a fixed cost term. The model is formulated as MINLP, where the average demand and total variance served by the DCs are calculated as the sum of the average demands and variances of the clients assigned to them, respectively. These average demands of the DCs are incorporated directly into the objective function through the economic order quantity (EOQ) expression, which in turn structures the working inventory costs. The variances of demand give the expression of the safety stock costs. It should be noted that they consider the ratio between the average demand and the variance of all customers constant, which simplifies the resolution of the problem. The authors propose a Lagrangian relaxation solution algorithm, in which they relax the restrictions of allocation customers to DCs. Shen et al. [12] restructure the model of Daskin et al. [11] as an IP model of Set-Covering; then they solve though branch-and-price approach, a variant of branch-and-bound in which nodes are processed by solving linear relaxations through column generation. Shu et al. [13] modify the model of Shen et al. [12], incorporating a generalization of the assumption that the demands and variances of the clients are proportional, making it more realistic. Similar to Shen et al. [12], they first restructure the model as a Set-Covering problem and solve it with the branch and price method, but making it more efficient. Snyder et al. [14] present a stochastic programming version of the Daskin et al. [11] model, where allocation decisions are made under random parameters such as the average daily demand and variance of the average demand of each retailer, which are described by discrete scenarios. The model minimizes the total expected cost (including location, transportation, and inventory costs) of the system in all scenarios. The location model explicitly handles the effects of economies of scale and risk pooling that result from the consolidation of inventory sites. They present an algorithm based on Lagrangian relaxation, which, as Daskin et al. [11] and Shen et al. [12], relaxes allocation constraints.

Miranda and Garrido [15] solve SCND through a simultaneous approach and incorporate inventory control decisions (EOQ and safety stock) within a CFLP, considering a stochastic demand distributed in a normal form, also modeling the phenomenon of risk pooling. This MINLP model is called a distribution network design model with risk pooling (DNDRP). The DNDRP includes, as constraints, the calculation of the total demands and variances served by each DC. This contrasts with the formulation of Daskin et al. [11], Shen et al. [12], Snyder et al. [14], and Ozsen et al. [16, 17], which incorporate them directly into the objective function through operation inventory costs and safety stock costs, respectively. Another difference between the models mentioned above is that Miranda and Garrido [15] do not explicitly consider economies of scale in transport costs. The deterministic capacity constraint of the DCs is formulated as described by Daskin [6]. The authors do not consider any assumption that may restrict the relationship between customer demands and variances.

The traditional deterministic capacitated location models do not consider inventory decision, and therefore capacity is typically calculated in an exogenous manner. As a result, to count enough inventory capacity, additional DCs must be installed. However, by ordering more frequently, we could have a lower average stock level and therefore lower costs. The papers that most resemble our work are the CFLP with stochastic inventory capacity and risk pooling proposed by Miranda and Garrido [18, 19] and Ozsen et al. [16, 17]; however, we consider a periodic review inventory control policy. Miranda and Garrido [18] use the same framework introduced in Miranda and Garrido [15] replacing the deterministic inventory capacity constraint in DCs by a stochastic constraint based on chance constrained programming. This constraint ensures that the inventory capacity for each DC is at least with respect to one $1\text{-}\beta$ probability. Additionally, they incorporate an order quantity restriction for each DC. One of the relevant conclusions of the modeling approach that they propose is that a decrease in the inventory capacity does not certainly imply an increase in the number of opened warehouses. In fact, decreasing the order size allows the optimal allocation of customers (those with more significant variances) in different warehouses, reducing the total cost of the system. Miranda and Garrido [19] use the same formulation of Miranda and Garrido [18]; nevertheless, the authors explain in detail the exact method of resolution to find solutions to the subproblems of each warehouse. This procedure is based on the incorporation of a constraint that represents a set of inequalities valid for D_i and V_i, where Ω is the domain of all the possible values of each combination of clients. The authors present a heuristic approach based on Lagrangian relaxation and the subgradient method. They relax the demand and variance constraints of DCs and allocation constraints. Lagos et al. [20] consider the Miranda and Garrido [18] model and solve it using a hybrid algorithm combining Ant Colony Optimization (ACO) and Lagrangian relaxation. They use ACO to assign clients to a subset of stores that is previously generated by Lagrangian relaxation. The results show that the hybrid approach is quite competitive, obtaining almost optimal solutions within a reasonable time.

The study by Ozsen et al. [16] is based on the model of Daskin et al. [11] to formulate a capacitated location model with risk pooling (CLMRP). The model captures the interdependence between capacity and inventory management in DCs. They assume that there is no correlation between daily retailer demands and that it follows a Poisson process [11, 12, 14]. This implies that the variance of the daily demand is equal to the daily demand average for each retailer. The model simultaneously determines warehouse locations, order sizes from the plant to warehouses, working and safety stock levels at warehouses, and the allocation of retailers to the warehouses. Similar to Miranda and Garrido [18, 19], the inventory capacity constraint is stochastically modeled by chance constrained programming. The authors propose a Lagrangian relaxation solution algorithm, in which they relax the allocation constraints, offering low gaps with moderate computational requirements for large-scale instances. Ozsen et al. [17] slightly modify the formulation developed by Oszen et al. [16], allowing retailers to be supplied by more than one DC on a probabilistic basis.

Jin et al. [21] propose a simultaneous localization and inventory model with multiple products. The model is formulated as the Capacitated P-Median Problem (CPMP). They assume that the stochastic demands of retailers are normally distributed. The model is formulated as a MINLP and solved through a combined simulation annealing algorithm (CSA). Chen et al. [22] discuss a reliable ILP, where facilities are subject to disruption risks. When a facility fails, customers can be reassigned to a different facility that exists to avoid high costs associated with loss of services. They propose a MINLP that minimizes the sum of installation costs, expected inventory costs, and costs expected under normal and breakdown states. They develop a Lagrangian relaxation solution framework, including an exact algorithm for relaxed nonlinear subproblems.

Several recent studies, including Atamtürk et al. [23], Shahabi et al. [24], and Schuster and Tancrez [25], have reformulated ILP with uncertain demand as Conic Quadratic Mixed-Integer Program (CQMIP). Atamtürk et al. [23] propose a joint inventory location model with stochastic demand considering various cases with uncapacitated and capacitated facilities, correlated retailer demand, stochastic lead times, and multiple products. Later, Shahabi et al. [24] study a location problem with a three-level inventory, where the demand for retailers is assumed to be correlated. Besides, they propose a solution approach, based on an external approximation algorithm, which shows the advantage of using this methodology. Finally, the authors show that the omission of the effect of correlation can lead to substantially suboptimal solutions. Schuster and Tancrez [25] provide a nonlinear continuous formulation that integrates location, order, inventory, and assignment decisions and includes transport, cycle, and safety stock costs. Then, considering that the model becomes linear when specific variables are fixed, they propose a heuristic algorithm that solves the resulting linear program. Finally, they use the solution to improve the estimates of variables for the next iteration. In order to show the efficiency of the algorithm, they compare their results with those of Atamtürk et al., 2012 [23]. They conclude that safety stock and risk pooling in retailers affect the design of a supply chain.

Petridis [26] addresses the optimal design of a multiproduct and multistep supply network under demand uncertainty. The system consists of multiproduct production sites, warehouses, and distribution centers and decisions are made regarding the selection of facilities and their capacity. Also, decision variables are based on the flow of products transferred and safety stock in each distribution center. The delivery time of an order to a customer is calculated, using the probabilities of excess and deficit of inventory. All these decisions are incorporated in a single period, configuring a MINLP. The author explores linearization techniques for the highly nonlinear terms selected from the models, reducing the computational effort for the solution of the model. Qu et al. [27] propose an ILP with stochastic demand through the application of two replacement policies, joint replenishment (JR) and independent replacement (IR). They solve the problem through three algorithms: Genetic Algorithm (GA), Evolutionary Differential Hybrid Algorithm (HDE), and Hybrid Self-adapting Evolutionary Differential Hybrid Algorithm (HSDE). Their computational results show the effectiveness of these algorithms. The results of the ILP suggest that the policy of JR can obtain better solutions regarding costs than the IR policy, due to the fixed ordering costs being shared in the same order.

All the previous papers and their associated analyzed models tended to focus on the ILP with inventory continuous review policy (s, S), rather than inventory periodic review policy. Yao et al. [28] discuss the latter of the two. They study a problem of location and inventory that incorporates multiple sources of warehouses, similar to that of Ozsen et al. [17]. In this problem, the multiple products are produced in several plants. The problem is formulated as a MINLP model. Berman et al. [29] incorporate a (R, S) periodic review inventory policy in the formulation of a coordinated inventory location model, where the choice of revision intervals in the DCs achieves coordination of the system. They present two types of coordination: total coordination, where all DCs have the same interval of review, and partial coordination, where each DC can choose its own review interval. While total coordination increases location costs and inventory costs, it is likely to reduce overall system operating costs, i.e., if operational costs such as scheduling delivery are taken into account. The problem is determining the location of the DCs, the allocation of retailers to the DCs, and the parameters of the inventory policy of the DCs, so that the total cost of the whole system is minimized. The model is formulated as a nonlinear integer programming problem and they solve it through an efficient Lagrangian relaxation algorithm. The results of their computational experiments and case study suggest that the increased costs due to full coordination, compared to partial coordination, are not significant. Therefore, total coordination, while making the model more practical, is economically justifiable. Cabrera et al. [30] formulate a novel joint localization and inventory model including a stochastic capacity constraint based on an Inventory Location Model Periodic Review (ILM-PR) inventory control policy. One of the modifications that they make regarding the continuous review policy is the incorporation of the undershoot concept that has not been considered in the previous ILP models. Based on this, they design a distribution network for a two-tier supply chain, quantifying the impact of the inventory control period review on the configuration costs of network and system. They do this considering both warehouse location and customer allocation decisions. To solve the problem they apply two heuristics, Tabu Search and Particle Swarm Optimization (PSO). According to the authors, this methodology shows an effective convergence rate. This

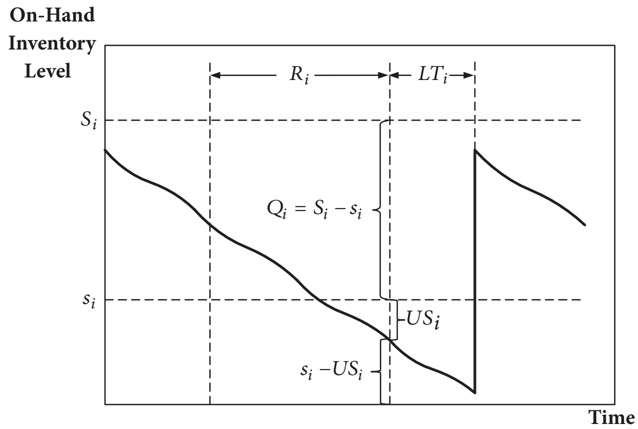

Figure 2: Inventory levels under an (s, S, R) control policy.

confirms that inventory control policy decisions have an effect on the design of the distribution network. Vahdani et al. [31] consider an ILP in a three-tier supply chain, where it is assumed that retailer demand is correlated and inventory shortage is allowed. The inventory periodic review control policy is utilized. In order to solve the joint ILP, they propose an optimization model based on MINLP, where the objective function is the minimization of total costs of the supply chain. To solve this MINLP model, they present a GA and a simulated annealing (SA) algorithm. Since the performance of the metaheuristic algorithms depends on the configuration of the parameters, the Taguchi method is used to establish the parameters of the indicated algorithms. Finally, the algorithms proposed by the authors are used in several numerical instances that indicate a better GA performance compared to the SA.

3. Inventory Control Policy and Total System Cost

In this section, we discuss inventory control and capacity constraint issues involved in a periodic review policy within the facility location modeling structure with stochastic demand. We will use the methodology proposed by Miranda and Cabrera [32] and Cabrera et al. [30]. When a periodic review is taken into account in an (s_i, S_i, R_i) inventory control policy, capacity constraints cannot be stated at any moment. In an (s_i, S_i, R_i) inventory control policy, inventory levels are reviewed after R_i periods for each warehouse i. Note that this parameter could be optimized; however, in the present research, it is fixed. In addition, if the inventory level is lower than the level s_i, then an order is placed to reach the objective level S_i. Consequently, order size for each warehouse i must consider the well-known undershoot magnitude (US_i), which is the number of items required to be ordered in addition to S_i-s_i, in order to reach S_i units of inventory, as shown in Figure 2. In other words, the US_i is the difference between the reorder point s_i and the inventory level directly prior to ordering.

For a given review period R_i, demand mean, and variance of a warehouse i (D_i and V_i), the average undershoot magnitude is computed as follows [33]:

$$US_i(D_i, V_i) = \frac{V_i}{2 \cdot D_i} + \frac{D_i R_i}{2} \tag{1}$$

Peak inventory levels are not controlled at any moment, solely in specific moments for each review period. This peak inventory level is reached only when orders arrive at the warehouse, LT_i time units after the previous order, and only if an order was submitted to the central warehouse or plant. Accordingly, each time an order arrives at a warehouse the inventory level is

$$\begin{aligned} &(s_i - US_i) + (S_i - s_i + US_i) - SD_i(LT_i) \\ &\quad = S_i - SD_i(LT_i) \end{aligned} \tag{2}$$

When an order is submitted to the plant, it is required that total inventory position reaches the level S_i, and LT_i later; inventory level is reduced by lead time demand $SD_i(LT_i)$. Similar to Miranda and Garrido [18, 19], we propose that this inventory capacity constraint must be reviewed for each peak inventory instant (i.e., for each order period) with a fixed and known probability$1-\beta$, but now assuming a periodic review, as follows:

$$\Pr(S_i - SD_i(LT_i) \le ICap) = 1 - \beta \tag{3}$$

This constraint is reformulated as a deterministic nonlinear constraint, which guarantees that the probabilistic constraint is fulfilled:

$$S_i \le ICap + D_i \cdot LT_i - Z_{1-\beta}\sqrt{V_i \cdot LT_i} \tag{4}$$

We specify the minimum order size as Q_i:

$$S_i = s_i + Q_i \longleftrightarrow Q_i = S_i - s_i \tag{5}$$

In consequence, constraint (4) can be written as

$$Q_i + s_i \le ICap + D_i \cdot LT_i - Z_{1-\beta}\sqrt{V_i \cdot LT_i} \tag{6}$$

Finally, the reorder point s_i is set in order to ensure that an order is not submitted at each moment in time (i.e., inventory level is larger than s_i). The inventory level must be enough to fill demand until the next order has arrived $R_i + LT_i$ time units, with a probability or service level $1-\alpha$:

$$\Pr\left(SD_i\left(R_i + LT_i\right) \le s_i\right) = 1 - \alpha \tag{7}$$

Similar to (3), this constraint is reformulated as a deterministic nonlinear constraint:

$$s_i = D_i \cdot \left(LT_i + R_i\right) + Z_{1-\alpha} \cdot \sqrt{LT_i + R_i}\sqrt{V_i} \tag{8}$$

Finally, replacing (8) in (6), the inventory capacity constraint for each warehouse i can be written as

$$Q_i + D_i R_i + \left(Z_{1-\alpha}\sqrt{LT_i + R_i} + Z_{1-\beta}\sqrt{LT_i}\right)\sqrt{V_i} \le ICap \tag{9}$$

Based on a periodic (s_i, S_i, R_i) inventory control policy, the safety stock to be included in the objective function is the average inventory level just before an order arrives at the warehouse:

$$\left(s_i - US_i\right) - D_i LT_i = D_i R_i + Z_{1-\alpha} \cdot \sqrt{LT_i + R_i}\sqrt{V_i} - US_i\left(D_i, V_i\right) \tag{10}$$

In addition, expected inventory and ordering costs related to order quantity or cycle inventory are evaluated in terms of the minimum order quantity Q_i and the average undershoot US_i, as in EOQ model:

$$\frac{OC_i \cdot D_i}{\left(Q_i + US_i\left(D_i, V_i\right)\right)} + \frac{HC_i \cdot \left(Q_i + US_i\left(D_i, V_i\right)\right)}{2} \tag{11}$$

4. Model Formulation

In this section, according to the previous inventory control assumptions, the Inventory Location Model with Stochastic Constraints of Inventory Capacity under Periodic Review (ILM-SCC-PR) is presented as a Stochastic Non-Linear Non-Convex Mixed Integer Programming (SNL-MIP) model. In this model, we tackle the problem of storage and delivery of a single product from a single plant or central warehouse to a collection of retailers through a set of candidate warehouses while minimizing the total system cost.

The parameters of the model are as follows:

N: number of available sites to install warehouses

M: number of customers to be served

RC_i: transportation unit cost between the plant and the warehouse i (\$/unit)

TC_{ij}: fixed transportation cost between the warehouse i and the customer j

F_i: operating fixed cost for each warehouse i (\$/day)

HC_i: holding cost per time unit at site i (\$/day)
OC_i: fixed ordering cost per time unit at site i (\$/day)

LT_i: deterministic lead time when ordering from warehouse i

d_j: mean of the daily demand for each customer j

v_j: variance of the daily demand for each customer j

v_j: variance of the daily demand for each customer j

$Z_{1-\alpha}$: value of the standard normal distribution, which accumulates a probability of $1-\alpha$

$Z_{1-\beta}$: value of the standard normal distribution, which accumulates a probability of $1-\beta$

$QCap_i$: order capacity of the warehouse i

$ICap_i$: inventory capacity of the warehouse i

The variables considered in the mathematical formulation are as follows:

X_i: it takes the value 1, if a warehouse is located on site i, and 0 otherwise

Y_{ij}: it takes the value 1, if warehouse i serves customer j, and 0 otherwise

Q_i: order size at the warehouse i (units)

D_i: served demand by each warehouse i (units)

V_i: variance of the served demand by each warehouse i

Consequently, the SNL-MIP model to solve the problem is

$$\begin{aligned}\min \quad & \sum_{i=1}^{N} F_i X_i + \sum_{i=1}^{N}\sum_{j=1}^{M}\left(RC_i d_j + TC_{ij}\right) Y_{ij} + \sum_{i=1}^{N}\left(OC_i \frac{D_i}{Q_i + US_i\left(D_i, V_i\right)} + HC_i \frac{Q_i + US_i\left(D_i, V_i\right)}{2}\right) \\ & + \sum_{i=1}^{N} HC_i\left(D_i R_i + Z_{1-\alpha}\sqrt{LT_i + R_i}\sqrt{V_i} - US_i\left(D_i, V_i\right)\right)\end{aligned} \tag{12}$$

$$\text{s.t.:} \quad \sum_{i=1}^{N} Y_{ij} = 1 \quad \forall j = 1, \ldots, M \tag{13}$$

$$Y_{ij} \le X_i \quad \forall i = 1, \ldots, N, \ \forall j = 1, \ldots, M \tag{14}$$

$$Q_i + D_i R_i + \left(Z_{1-\alpha}\sqrt{LT_i + R_i} + Z_{1-\beta}\sqrt{LT_i}\right)\sqrt{V_i} \le ICap_i \cdot X_i \quad \forall i = 1, \ldots, N \tag{15}$$

$$Q_i + US_i(D_i, V_i) \leq QCap_i \quad \forall i = 1, \ldots, N \tag{16}$$

$$D_i = \sum_{j=1}^{M} d_j Y_{ij} \quad \forall i = 1, \ldots, N \tag{17}$$

$$V_i = \sum_{j=1}^{M} v_j Y_{ij} \quad \forall i = 1, \ldots, N \tag{18}$$

$$US_i(D_i, V_i) = \frac{V_i}{2 \cdot D_i} + \frac{D_i R_i}{2} \quad \forall i = 1, \ldots, N \tag{19}$$

$$X_i, Y_{ij} \in \{0, 1\} \quad \forall i = 1, \ldots, N, \ \forall j = 1, \ldots, M \tag{20}$$

The objective function (12) minimizes the total system cost. The first term is the fixed and operating costs when opening warehouses. The second term is the transportation cost between each warehouse and its allocated customers, plus the transportation and ordering costs between the plant and warehouses. The third term contains fixed and inventory costs related to warehouse order size. The fourth term represents the storage cost associated with safety stock at each warehouse. Constraints (13) ensure that each customer is served exactly by one warehouse. Constraints (14) state that customers can only be assigned to open warehouses ($X_i = 1$). Constraints (15) ensure that inventory capacity for each warehouse is fulfilled at least with a probability $1 - \beta$. Constraints (16) ensure that the order size is below the capacity order size allowed to warehouse i. Equations (17) and (18) determine the mean and variance of the served demand by each warehouse. Equations (19) calculate average undershoot magnitude for each warehouse. Finally, (20) indicates the domain of decision variables.

The objective function and the two stochastic constraints are nonlinear, resulting in a model that is very hard to solve for large-scale instances. The complexity of the problem motivated us to propose a heuristic approach to solve it. An explanation of the algorithm is described in the next section.

5. Solution Approach

Most of the conventional location models have been solved successfully by Lagrangian relaxation-based heuristics. Fisher [34, 35] provides a detailed analysis of Lagrangian relaxation. Likewise, Daskin [6] applies the same solution approach to solve the UFLP and the CFLP obtaining reasonably good results. Because ILM-SCC-PR is an extension of the UFLP, we implement a Lagrangian relaxation algorithm and subgradient method to solve it. We develop two relaxations to solve the ILM-SCC-PR. First, we relax constraints (17) and (18), decoupling binary network design variables (X and Y) from inventory control decisions (Q) and mean and variance for demand (D and V) in each warehouse. In addition, we relax customer assignment constraints (13), similar to several Lagrangian relaxation applications for standard FLP and ILP. Second, we relax only constraints (17) and (18).

5.1. First Lagrangian Relaxation Algorithm. Associating the dual variables vectors λ and ω with the constraints (17) and (18), respectively, and ψ with constraint (13), we obtain the following relaxed problem:

RLP_1

$$\begin{aligned}
\min \quad & \sum_{i=1}^{N} F_i X_i + \sum_{i=1}^{N}\sum_{j=1}^{M} \left(\left(RC_i + \lambda_i \right) d_j + TC_{ij} + \omega_i v_j - \psi_j \right) Y_{ij} + \sum_{i=1}^{N} \left(OC_i \frac{D_i}{\left(Q_i + US_i(D_i, V_i) \right)} + HC_i \frac{\left(Q_i + US_i(D_i, V_i) \right)}{2} \right) \\
& + \sum_{i=1}^{N} HC_i \left(D_i R_i + Z_{1-\alpha} \sqrt{LT_i + R_i} \sqrt{V_i} - US_i(D_i, V_i) \right) - \sum_{i=1}^{N} \left(\lambda_i D_i + \omega_i V_i \right) + \sum_{j=1}^{M} \psi_j \\
\text{s.t.:} \quad & Y_{ij} \leq X_i \quad \forall i = 1, \ldots, N, \ \forall j = 1, \ldots, M \\
& Q_i + D_i R_i + \left(Z_{1-\alpha} \sqrt{LT_i + R_i} + Z_{1-\beta} \sqrt{LT_i} \right) \sqrt{V_i} \leq ICap_i \cdot X_i \quad \forall i = 1, \ldots, N \\
& Q_i + US_i(D_i, V_i) \leq QCap_i \quad \forall i = 1, \ldots, N \\
& US_i(D_i, V_i) = \frac{V_i}{2 \cdot D_i} + \frac{D_i R_i}{2} \quad \forall i = 1, \ldots, N \\
& X_i, Y_{ij} \in \{0, 1\} \quad \forall i = 1, \ldots, N, \ \forall j = 1, \ldots, M
\end{aligned} \tag{21}$$

For fixed values of the Lagrangian multipliers, λ, ω, and ψ, we minimize (21) over location variables, X_i, and the assignment variables Y_{ij}. For the given λ, ω, and ψ vectors, the problem decouples to the following subproblem for each warehouse i:

SP_i

$$\begin{aligned}
\min \quad & F_i X_i + \sum_{j=1}^{M} \left(\left(RC_i + \lambda_i \right) d_j + TC_{ij} + \omega_i v_j - \psi_j \right) Y_{ij} + \left(OC_i \frac{D_i}{\left(Q_i + US_i \left(D_i, V_i \right) \right)} + HC_i \frac{\left(Q_i + US_i \left(D_i, V_i \right) \right)}{2} \right) \\
& + HC_i \left(D_i R_i + Z_{1-\alpha} \sqrt{LT_i + R_i} \sqrt{V_i} - US_i \left(D_i, V_i \right) \right) - \left(\lambda_i D_i + \omega_i V_i \right) \\
\text{s.t.:} \quad & Y_{ij} \le X_i \quad \forall j = 1, \ldots, M \\
& Q_i + D_i R_i + \left(Z_{1-\alpha} \sqrt{LT_i + R_i} + Z_{1-\beta} \sqrt{LT_i} \right) \sqrt{V_i} \le ICap_i \cdot X_i \\
& Q_i + US_i \left(D_i, V_i \right) \le QCap_i \\
& US_i \left(D_i, V_i \right) = \frac{V_i}{2 \cdot D_i} + \frac{D_i R_i}{2} \\
& \left(D_i, V_i \right) \in \Omega \\
& X_i, Y_{ij} \in \{0, 1\} \quad \forall j = 1, \ldots, M
\end{aligned} \tag{22}$$

We include a set of valid inequalities $(D_i, V_i) \in \Omega$ to solve previous subproblems and to reduce duality gaps by increasing upper bounds. Valid inequalities are defined as a set of constraints, which bound all feasible solutions of dependent variables D_i and V_i [19].

Each subproblem (22) may be decoupled for the fixed values of the Lagrangian multipliers for each iteration k, $\lambda_i^k, \omega_i^k, \psi_i^k$, as follows:

SP_i^{1k}

$$\begin{aligned}
\Pi_i^k = \min \quad & \left(OC_i \frac{D_i}{\left(Q_i + US_i \left(D_i, V_i \right) \right)} + HC_i \frac{\left(Q_i + US_i \left(D_i, V_i \right) \right)}{2} \right) + HC_i \left(D_i R_i + Z_{1-\alpha} \sqrt{LT_i + R_i} \sqrt{V_i} - US_i \left(D_i, V_i \right) \right) \\
& - \left(\lambda_i^k D_i + \omega_i^k V_i \right) \\
\text{s.t.:} \quad & Q_i + D_i R_i + \left(Z_{1-\alpha} \sqrt{LT_i + R_i} + Z_{1-\beta} \sqrt{LT_i} \right) \sqrt{V_i} \le ICap_i \\
& Q_i + US_i \left(D_i, V_i \right) \le QCap_i \\
& US_i \left(D_i, V_i \right) = \frac{V_i}{2 \cdot D_i} + \frac{D_i R_i}{2} \\
& \left(D_i, V_i \right) \in \Omega
\end{aligned} \tag{23}$$

SP_i^{2k}

$$\begin{aligned}
\theta_i^k = \min \quad & \left(F_i + \Pi_i \right) X_i \\
& + \sum_{j=1}^{M} \left(\left(RC_i + \lambda_i^k \right) d_j + TC_{ij} + \omega_i^k v_j - \psi_j^k \right) Y_{ij} \\
\text{s.t.:} \quad & Y_{ij} \le X_i \quad \forall j = 1, \ldots, M \\
& X_i, Y_{ij} \in \{0, 1\} \quad \forall j = 1, \ldots, M
\end{aligned} \tag{24}$$

θ_i denotes the benefit of facility i and represents the contribution of opening facility i to the objective function (12). This decomposition consists of solving SP_i^1 to compute Π_i and then solving SP_i^2 to calculate θ_i, based on the computed Π_i, as explained in Section 5.3.

5.2. Second Lagrangian Relaxation Algorithm. Associating the dual variables vectors λ and ω with constraints (17) and (18), respectively, we obtain the following relaxed problem:

RLP_2

$$\begin{aligned}
\min \quad & \sum_{i=1}^{N} F_i X_i + \sum_{i=1}^{N}\sum_{j=1}^{M} \left(\left(RC_i + \lambda_i \right) d_j + TC_{ij} + \omega_i v_j \right) Y_{ij} + \sum_{i=1}^{N} \left(OC_i \frac{D_i}{\left(Q_i + US_i \left(D_i, V_i \right) \right)} + HC_i \frac{\left(Q_i + US_i \left(D_i, V_i \right) \right)}{2} \right) \\
& + \sum_{i=1}^{N} HC_i \left(D_i R_i + Z_{1-\alpha} \sqrt{LT_i + R_i} \sqrt{V_i} - US_i \left(D_i, V_i \right) \right) - \sum_{i=1}^{N} \left(\lambda_i D_i + \omega_i V_i \right) \\
\text{s.t.:} \quad & \sum_{i=1}^{N} Y_{ij} = 1 \quad \forall j = 1, \ldots, M \\
& Y_{ij} \leq X_i \quad \forall i = 1, \ldots, N, \ \forall j = 1, \ldots, M \\
& Q_i + D_i R_i + \left(Z_{1-\alpha} \sqrt{LT_i + R_i} + Z_{1-\beta} \sqrt{LT_i} \right) \sqrt{V_i} \leq ICap \cdot X_i \quad \forall i = 1, \ldots, N \\
& Q_i + US_i \left(D_i, V_i \right) \leq QCap \quad \forall i = 1, \ldots, N \\
& US_i \left(D_i, V_i \right) = \frac{V_i}{2 \cdot D_i} + \frac{D_i R_i}{2} \quad \forall i = 1, \ldots, N \\
& X_i, Y_{ij} \in \{0, 1\} \quad \forall i = 1, \ldots, N, \ \forall j = 1, \ldots, M
\end{aligned} \tag{25}$$

For fixed values of the Lagrangian multipliers, λ and ω, we want to minimize (25) over location variables, X_i, and the assignment variables Y_{ij}. For the given λ and ω vectors, the problem decouples to the following subproblems:

$SP1$

$$\begin{aligned}
\min \quad & \sum_{i=1}^{N} \left(OC_i \frac{D_i}{\left(Q_i + US_i \left(D_i, V_i \right) \right)} + HC_i \frac{\left(Q_i + US_i \left(D_i, V_i \right) \right)}{2} \right) + \sum_{i=1}^{N} HC_i \left(D_i R_i + Z_{1-\alpha} \sqrt{LT_i + R_i} \sqrt{V_i} - US_i \left(D_i, V_i \right) \right) \\
& - \sum_{i=1}^{N} \left(\lambda_i D_i + \omega_i V_i \right) \\
\text{s.t.:} \quad & Q_i + D_i R_i + \left(Z_{1-\alpha} \sqrt{LT_i + R_i} + Z_{1-\beta} \sqrt{LT_i} \right) \sqrt{V_i} \leq ICap \cdot X_i \quad \forall i = 1, \ldots, N \\
& Q_i + US_i \left(D_i, V_i \right) \leq QCap \quad \forall i = 1, \ldots, N \\
& US_i \left(D_i, V_i \right) = \frac{V_i}{2 \cdot D_i} + \frac{D_i R_i}{2} \quad \forall i = 1, \ldots, N
\end{aligned} \tag{26}$$

$SP2$

$$\begin{aligned}
\min \quad & \sum_{i=1}^{N} F_i X_i + \sum_{i=1}^{N}\sum_{j=1}^{M} \left(\left(RC_i + \lambda_i \right) d_j + TC_{ij} + \omega_i v_j \right) Y_{ij} \\
\text{s.t.:} \quad & \sum_{i=1}^{N} Y_{ij} = 1 \quad \forall j = 1, \ldots, M \\
& Y_{ij} \leq X_i \quad \forall i = 1, \ldots, N, \ \forall j = 1, \ldots, M \\
& X_i, Y_{ij} \in \{0, 1\} \quad \forall i = 1, \ldots, N, \ \forall j = 1, \ldots, M
\end{aligned} \tag{27}$$

Each subproblem (26) may be decoupled for the fixed values of the Lagrangian multipliers for each iteration k, λ_i^k, ω_i^k, in the following subproblems for each warehouse i:

$SP1_i^k$

$$\begin{aligned}
\Pi_i^k = \min \quad & \left(OC_i \frac{D_i}{\left(Q_i + US_i\left(D_i, V_i\right)\right)} + HC_i \frac{\left(Q_i + US_i\left(D_i, V_i\right)\right)}{2} \right) + HC_i \left(D_i R_i + Z_{1-\alpha} \sqrt{LT_i + R_i} \sqrt{V_i} - US_i\left(D_i, V_i\right) \right) \\
& - \left(\lambda_i^k D_i + \omega_i^k V_i \right) \\
\text{s.t.:} \quad & Q_i + D_i R_i + \left(Z_{1-\alpha} \sqrt{LT_i + R_i} + Z_{1-\beta} \sqrt{LT_i} \right) \sqrt{V_i} \leq ICap \\
& Q_i + US_i\left(D_i, V_i\right) \leq QCap \\
& US_i\left(D_i, V_i\right) = \frac{V_i}{2 \cdot D_i} + \frac{D_i R_i}{2} \\
& \left(D_i, V_i\right) \in \Omega
\end{aligned} \tag{28}$$

$SP2^k$

$$\begin{aligned}
\theta^k = \min \quad & \sum_{i=1}^{N} \left(F_i + \Pi_i^k \right) X_i + \sum_{i=1}^{N} \sum_{j=1}^{M} \left(\left(RC_i + \lambda_i^k \right) d_j + TC_{ij} + \omega_i^k v_j \right) Y_{ij} \\
\text{s.t :} \quad & \sum_{i=1}^{N} Y_{ij} = 1 \quad \forall j = 1, \dots, M \\
& Y_{ij} \leq X_i \quad \forall i = 1, \dots, N, \ \forall j = 1, \dots, M \\
& X_i, Y_{ij} \in \{0, 1\} \quad \forall i = 1, \dots, N, \ \forall j = 1, \dots, M
\end{aligned} \tag{29}$$

This decomposition consists of solving $SP1_i$ to compute Π_i and then solving $SP2$ to calculate θ, based on the computed Π_i, as explained in Section 5.3.

5.3. Subproblem Solving

5.3.1. First Lagrangian Relaxation. For fixed values of the Lagrangian multipliers $\lambda_i^k, \omega_i^k, \psi_i^k$, which are associated with relaxing constraints (17), (18), and (13), respectively, we obtain an infeasible solution of the primal problem in each iteration k of the algorithm. This solution generates a lower bound on the optimal value of the primal problem.

First, we solve SP_i^1 to calculate the value of Π_i of subproblems (23), for which an exact procedure is found in Miranda [36]. Once Π_i is obtained, SP_i^2 is solved based on the value of Π_i according to Algorithm 1 (see Appendix A).

5.3.2. Second Lagrangian Relaxation. For fixed values of the Lagrangian multipliers λ_i^k and ω_i^k, which are associated with relaxing constraints (17) and (18), respectively, we obtain an infeasible solution of the primal problem in each iteration k of the algorithm. As in the first Lagrangian relaxation, this solution corresponds to a lower bound on the optimal value of the primal problem. First, we solve $SP1_i$ to calculate the value of Π_i of subproblems (26), which is identical to subproblem (23). Once Π_i is obtained, $SP2$ is solved based on the value of Π_i through the solver CPLEX.

5.4. Lagrangian Heuristic and Subgradient Optimization. At each iteration k of the Lagrangian algorithm, we use the current lower bound solution to obtain a feasible solution, which is an upper bound to the optimal value of the primal problem. The Lagrangian heuristic considers three main procedures: warehouse selection, greedy assignment of customers, and K-OPT improvements. These three procedures are run for different numbers of warehouses, from 1 to N, based on the results and dual information of the subproblems SP_i. Namely, the complete heuristic is executed N times, and the best solution is selected. Notice the high complexity of the heuristic, especially, K-OPT improvement procedure, in contrast to the standard, simple Lagrangian heuristic observed in the literature. In order to avoid a potential high time consumption, only the K-OPT procedure is executed every 30 iterations of the algorithm. The three main procedures are described as follows.

for $i = 1$ to $N\{$

Compute $\Delta_i = F_i + \Pi_i + \sum_{j=1}^{M} \min\{0, (RC_i + \lambda_i)d_j + TC_{ij} + \omega_i v_j - \psi_j\}$

If $\Delta_i < 0$ **then**

$X_i = 1$, and

$$Y_{ij} = \begin{cases} 1 & \text{if } (RC_i + \lambda_i)d_j + TC_{ij} + \omega_i v_j - \psi_j < 0 \\ 0 & \text{if } (RC_i + \lambda_i)d_j + TC_{ij} + \omega_i v_j - \psi_j \geq 0 \end{cases} \quad \forall j = 1, \ldots, M$$

D_i, V_i and Q_i retain the values computed from the resolution of subproblem (23)

If $\Delta_i \geq 0$ **then**

$X_i = D_i = V_i = Q_i = 0$,

$Y_{ij} = 0, \quad \forall j = 1, \ldots, M$

$\}$

Algorithm 1: Subproblem solving for first Lagrangian relaxation.

for $i = 1$ to N$\{$

$$\overline{\Pi}_i^k = \frac{OC_i \overline{D}_i^k}{(\overline{Q}_i^k + US_i(\overline{D}_i^k, \overline{V}_i^k))} + HC_i \frac{(\overline{Q}_i^k + US_i(\overline{D}_i^k, \overline{V}_i^k))}{2} + HC_i(\overline{D}_i^k R_i + Z_{1-\alpha}\sqrt{LT_i + R_i}\sqrt{\overline{V}_i^k} - US_i(\overline{D}_i^k, \overline{V}_i^k)) - \lambda_i^k \overline{D}_i^k - \omega_i^k \overline{V}_i^k$$

$$\overline{\Delta}_i = F_i + \overline{\Pi}_i^k + \sum_{j=1}^{M} \min\{0, (RC_i + \lambda_i^k)d_j + TC_{ij} + \omega_i^k v_j - \psi_j^k\}$$

$\}$

return (P sites in ascending order of $\overline{\Delta}_i$)

Algorithm 2: Warehouse selection algorithm.

5.4.1. Warehouse Selection. This procedure assumes that the optimal solution $\overline{x}^k = (\overline{X}^k, \overline{Y}^k, \overline{D}^k, \overline{V}^k, \overline{Q}^k)$ of the subproblems SP_i and the Lagrange multipliers $(\lambda_i^k, \omega_i^k, \psi_i^k)$ are known. For the warehouse selection, the optimal costs of subproblems SP_i are taken as initial values. Then, the best $P(\leq N)$ warehouses are chosen (see Algorithm 2 in Appendix A).

5.4.2. Greedy Assignment of Customers. Once the warehouses are chosen, the customers are greedy assigned to the chosen warehouses; i.e., each client is assigned to the nearest warehouse, based on the transportation cost $RC_i \cdot d_j + TC_{ij}$, respecting the constraint of ordering capacity and maximum inventory. In order to satisfy these constraints, we calculate three types of order quantities at each warehouse i. First is Q_i^{EOQ}, which is economic order quantity in absence of capacity constraints. Second is Q_i', which is the available inventory capacity once inventory associated with variances are discounted, based on the inventory capacity constraint. Third is Q_i'', which is the available order quantity once undershoot is subtracted, based on the order capacity constraint. Then, the optimal order quantity Q_i^* is the minimum of the three previous different values for Q_i, as long as Q_i^* has a nonnegative value. Otherwise, delete i from the potential site's pool. The heuristic is described according to Algorithm 3 (see Appendix A).

5.4.3. K-OPT Improvements. Once a feasible solution is obtained through the last two steps (i.e., $\tilde{x}^k = (\tilde{X}^k, \tilde{Y}^k, \tilde{D}^k, \tilde{V}^k, \tilde{Q}^k)$), two K-OPT improvements are run, 1-OPT and 2-OPT. The former evaluates the reassignment of each customer to the other installed warehouses, if capacity constraints allow it; then, the best feasible interchange is chosen. If the total cost decreases then the reassignment is permanent. The latter takes pairs of clients in different warehouses and swaps them if capacity constraints allow it. If the total cost decreases the swap becomes permanent. In this algorithm, the optimal value of the dual problem is obtained based on dual maximization, which represents a lower bound to the optimal value of the problem P. Thus, the difference between this lower bound and the cost of the best solution obtained through the heuristic previously described is an upper bound to errors of the heuristic solutions.

The update of dual variables in each iteration k is based on the subgradient method [37, 38]. This method employs the slackness/violation vector associated with relaxed constraints. Furthermore, this method utilizes an upper bound UB on the optimal value of the primal problem, which is obtained by solving the Lagrangian heuristic procedure described previously in this section. The procedure is repeated until a standard convergence criterion is met.

6. Numerical Results and Discussion

In this section, we study the quality of the solutions by the proposed heuristic procedure. Furthermore, we validate the model ILM-SCC-PR and its heuristic solutions. We used the instances of Miranda and Garrido [18, 19] as a benchmark

for $i = 1$ to N {
for $j = 1$ to M {
$dk_{ij} = RC_{I_i} \cdot d_j + TC_{I_i,j}$
}
}
for $q = 1$ to N{
for $i = 1$ to q{
$\widetilde{X}_{I_i} = 1$
}
for $j = \mathbf{1}$ to M{
for $i = 1$ to N{
$\widetilde{Y}_{J_{ij},j} = 1$
for $l = 1$ to N{
$\widetilde{D}_l = \sum_{s=1}^{m} d_s \cdot \widetilde{Y}_{ls}$
$\widetilde{V}_l = \sum_{s=1}^{m} v_s \cdot \widetilde{Y}_{ls}$
}
$US_{J_{ij}} = \dfrac{\widetilde{V}_{J_{ij}}}{2} \cdot \widetilde{D}_{J_{ij}} + \dfrac{\widetilde{D}_{J_{ij}} R_{J_{ij}}}{2}$
$Q^{EOQ}_{J_{ij}} = \sqrt{\dfrac{(2 \cdot OC_{J_{ij}} \cdot \widetilde{D}_{J_{ij}})}{HC_{J_{ij}}}} - US_{J_{ij}}$
$Q'_{J_{ij}} = ICap - \widetilde{D}_{J_{ij}} R_{J_{ij}} - (Z_{1-\alpha}\sqrt{LT_{J_{ij}} + R_{J_{ij}}} + Z_{1-\beta}\sqrt{LT_{J_{ij}}})\sqrt{\widetilde{V}_{J_{ij}}}$
$Q''_{J_{ij}} = QCap - US_{J_{ij}}$
$Q^{*}_{J_{ij}} = \min(Q^{EOQ}_{J_{ij}}, Q'_{J_{ij}}, Q''_{J_{ij}})$
if $Q^{*}_{J_{ij}} > 0$ **then**
break i
else
$\widetilde{Y}_{J_{ij},j} = 0,\ US_{J_{ij}} = 0, Q^{*}_{J_{ij}} = 0$
next i
}
}
$\widetilde{x}^k = (\widetilde{X}^k, \widetilde{Y}^k, \widetilde{D}^k, \widetilde{V}^k, \widetilde{Q}^k)$
$UB^k = f(\widetilde{x}^k)$
$1 - OPT$
$2 - OPT$
}

Algorithm 3: Greedy assignment of customers and local search algorithm.

for an ILP under continuous review with the assumptions required for a periodic review problem.

We used an Intel Core i3 processor at 2.4 GHz with 6 GB of RAM and Windows 7 to run the heuristic procedure. The program was developed in Microsoft Visual Studio 2010 C++ and the subproblems of Lagrangian relaxation were solved in IBM CPLEX 12.5. The numerical experiments have 20 warehouses and 40 clients (840 binary variables). The main aim of presenting these experiments is to show the quality of the heuristic solutions in terms of their differences with the dual optimal values. This provides lower bounds for the optimal solution for the original problem. In addition, we test the performance with two different Lagrangian relaxations, as we explain previously in Sections 5.1 and 5.2, respectively. The average execution time for the test examples of the first and second Lagrangian relaxation was 42 and 102 seconds, respectively.

The model and the heuristic approach were validated through a sensitivity analysis of the following key parameters: ordering capacity, demand variability, and fixed costs. We considered two levels of order capacity: $QCap = 600$ and 900. Demand variances and warehouse fixed location costs ranged over seven values from the base case: $\pm 0\%, \pm 10\%, \pm 20\%$, and $\pm$ 30% each. Two values of the review period were considered: R=1, 3. A total of 2×7×7×2 = 196 instances were solved for each one of two Lagrangian relaxations, which sum to 196×2 = 392 instances finally.

The cost parameters are expressed in a generic cost unit, CU. Fixed costs F, ordering costs, OC, and lead times, LT, for each warehouse are reported in Table 2. For holding costs,

Table 1: Parameters for the Lagrangian relaxation procedure.

Parameter	Value
Maximum number of iterations	5000
Number of iterations before halving α	30
Initial value of α	2
Minimum value of α	0.0000001
Minimum LB-UB gap	0.001%
Initial value for Lagrangian multipliers	0.0

Table 2: Parameters of warehouses or distribution centers, W.

W	1	2	3	4	5	6	7	8	9	10
F	103,062	81,691	104,051	103,724	89,875	124,375	101,713	87,989	106,199	98,629
OC	61,800	47,150	41,940	88,650	62,100	55,220	41,470	62,650	68,440	69,080
LT	3	2	2	4	2	2	2	2	3	3
W	11	12	13	14	15	16	17	18	19	20
F	103,648	93,505	76,507	93,668	83,391	100,396	104,592	114,521	123,498	91,817
OC	64,070	45,320	69,690	45,680	77,260	41,000	74,780	53,030	32,930	76,990
LT	3	2	3	2	3	2	3	2	1	3

Table 3: Demand parameters of customers.

Customer	1	2	3	4	5	6	7	8	9	10
Mean	73.81	68.86	70.24	64.07	69.52	69.96	76.01	61.74	63.92	74.26
Variance	1,249.06	979.75	1,112.21	955.50	1,132.31	1,152.86	1,380.28	837.35	946.12	1,192.01
Customer	11	12	13	14	15	16	17	18	19	20
Mean	73.50	67.58	69.02	70.62	63.26	75.95	66.70	66.53	68.30	72.43
Variance	1,304.16	1,129.90	1,188.83	1,166.77	900.02	1,378.81	958.43	1,026.32	1,029.22	1,153.60
Customer	21	22	23	24	25	26	27	28	29	30
Mean	57.65	82.92	57.99	65.32	61.99	77.96	63.03	75.06	60.79	64.73
Variance	737.09	1,565.53	776.18	1,035.71	908.62	1,427.99	922.58	1,402.98	931.67	999.49
Customer	31	32	33	34	35	36	37	38	39	40
Mean	69.28	72.99	71.01	72.01	81.32	72.55	73.1	65.24	52.74	69.88
Variance	1,053.06	1,104.93	1,146.79	1,170.89	1,439.62	1,334.44	1,314.54	1,022.56	783.50	1,215.62

HC, and transportation costs, *RC*, a value of 100 *CU* was assumed. Also, *ICap is equal to 1200*. $Z_{1-\alpha}$ and $Z_{1-\beta}$ were set to be 1.64 (95% of service level).

The parameters for Lagrangian relaxation used for all experiments are given in Table 1. We determined the Lagrangian procedure based on the maximum number of iterations allowed, or the optimality gap, or the minimum value of α (the scale used in calculating the different step sizes for updating each Lagrange multiplier), whichever happened first. The optimality gap is defined as ((UB-LB)/LB) ×100.

The customer's mean and variance are shown in Table 3. Both the clients and potential warehouse sites were randomly distributed over a square with 2000 km sides. Transportation costs *TC* were assumed as 56 CU/km. For more details of *TC* complete data, see Tables 14 and 15 in Appendix C.

The upper bounds of errors were between 0.5% and 2.5%, and 0.5% and 3.0% for first and second relaxation, respectively, considering R=1, showing the quality of the found solutions. The histogram for the upper bounds of errors is shown in Figure 3. The average error obtained was 1.1% and 1.3% for first and second relaxation, correspondingly.

The upper bounds of errors were between 4.0% and 9.0%, and 5.0% and 9.0% for first and second relaxation, respectively, considering R=3, showing a worst quality of the found solutions comparatively with R=1 solutions. The histogram for the upper bounds of errors is shown in Figure 4. The average error obtained was 6.4% and 6.5% for first and second relaxation, correspondingly.

Table 4 shows the solutions obtained considering both values of ordering capacity (600 and 900), for variances at baseline and R=1 for first and second relaxations. It presents the installed warehouses (W), the served demands, and variance of the served demand by each warehouse (D and V, respectively). It also displays the optimal order

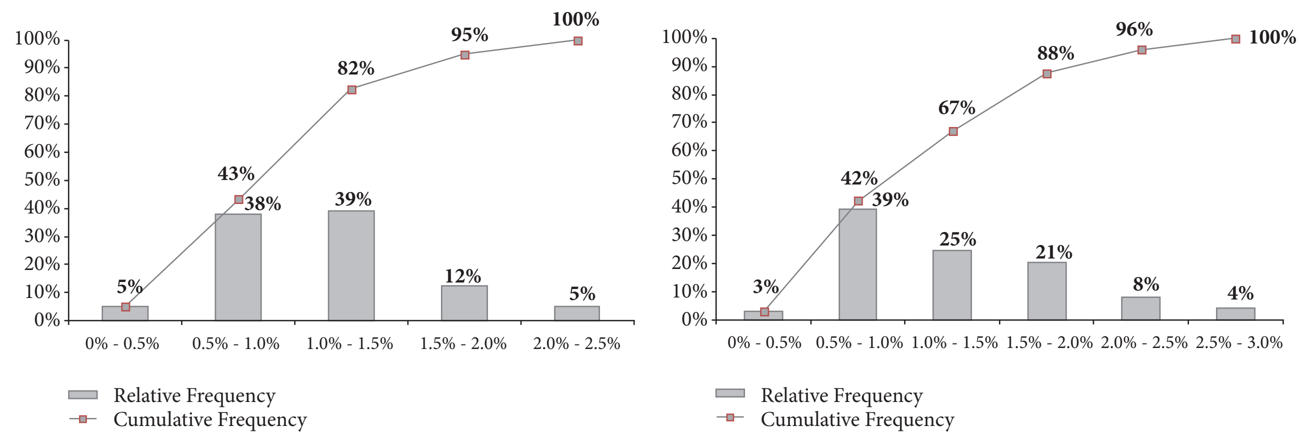

Figure 3: Observed upper bounds for the solution errors in the 98 analyzed instances for first and second relaxation, R=1.

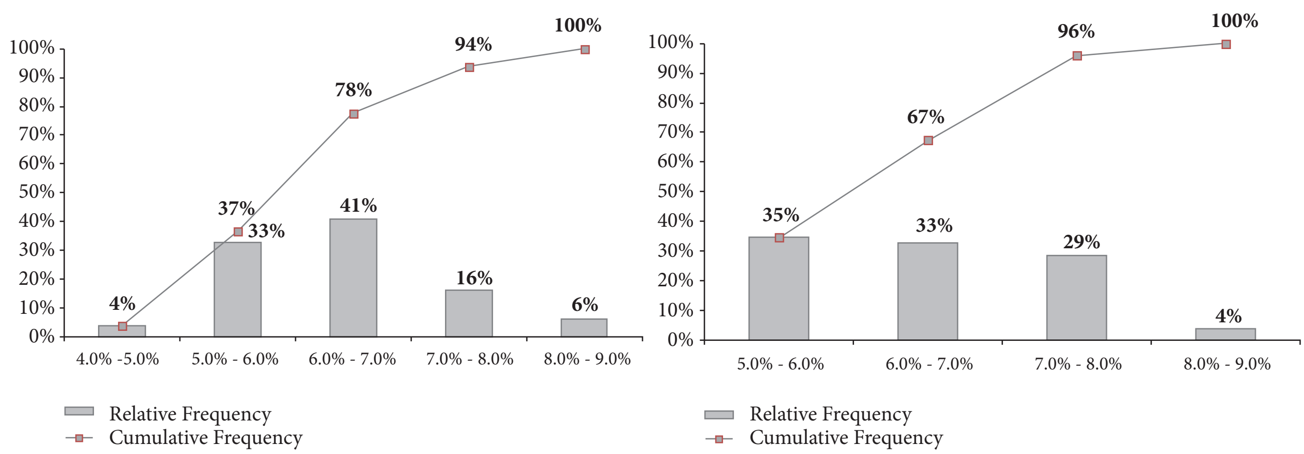

Figure 4: Observed upper bounds for the solution errors in the 49 analyzed instances for first and second relaxation, R=3.

quantity in absence of capacity constraints (Q^{EOQ}) and the available inventory capacity once the inventory associated with variances is discounted based on the inventory capacity constraint (Q'). It also shows the available order quantity once undershoot is subtracted based on the order capacity constraint (Q^*) and the order quantity given by the heuristic, Q^*. It can be noted that the order quantity given by the heuristic never violates the constraints and in all cases is the same as Q',which means that the inventory capacity constraints are active. Correspondingly, Table 5 presents the same outcomes but now considering a period of R=3. In this case, the order quantity additionally takes the same value of Q'; it can also be equal to Q^{EOQ}, which means that neither inventory nor order capacity constraints are active.

The details of the solutions of first and second relaxation are presented in Tables 6 and 7, respectively, in which the columns are as follows: *Prob no.:* problem number, *FC:* factor of fixed cost sensitivity (i.e., 0.7 corresponds to a variation -30%), FV: factor of variance sensitivity, *DCs opened:* the additional DCs that are located compared to the baseline instance, *DCs closed:* the additional DCs that are closed compared to the baseline instance. *No. of open DCs:* total number of DCs that are open, *Upper Bound:* objective value of the best feasible solution, *Lower Bound*: the best lower bound found for optimal objective function, % *Gap:* percentage gap between upper bound and lower bound solution, *Lag iter:* total number of Lagrangian relaxation iterations, and *CPU time (s):* the number of CPU seconds elapsed when the algorithm terminates.

Note that upper bound values tend to increase with respect to the increment of the fixed cost (*FC*). A similar behavior is observed for variation in demand variance (*FV*). Both tendencies denote a reasonable response of the Lagrangian heuristic since it is expected that system costs increase with respect to both sets of parameters. On the other hand, if we compare results in Table 6 for the first relaxation and results in Table 7 for the second relaxation increasing order capacity constraints a system cost reduction is produced. Finally, when we compare results in Table 6 for first relaxation and results in Table 7 for second relaxation, an increment in the duration of the review period (R = 1 and R = 3) produces worst solutions in terms of system cost and % Gap. These results show the reasonability of the Lagrangian heuristic, based on the tendencies of the objective function when different input parameters are modified (see Tables 8–13 in Appendix B for more details).

Table 4: Sensitivity analysis for the capacity constraints with variances at baseline and R=1.

First relaxation: $QCap$ = 600, 900							Second relaxation: $QCap$ = 600, 900						
W	D	V	Q^{EOQ}	Q'	Q''	Q^*	W	D	V	Q^{EOQ}	Q'	Q''	Q^*
2	661.9	10,433.7	451.2	11.1	261.2	11.1	2	661.9	10,433.7	451.2	11.1	561.2	11.1
3	618.2	9,990.7	402.9	66.0	282.8	66.0	3	614.0	9,862.1	402.6	73.6	585.0	73.6
5	565.7	9,332.3	547.1	135.9	308.9	135.9	5	496.7	8,143.5	528.9	237.7	643.5	237.7
8	486.9	7,865.7	529.6	255.5	348.5	255.5	8	562.8	9,244.5	550.2	141.1	610.4	141.1
14	421.2	6,845.0	401.6	351.9	381.3	351.9	10	418.5	6,783.6	543.1	277.4	682.6	277.4

Table 5: Sensitivity analysis for the capacity constraints with variances at baseline and R=3.

First relaxation: $QCap$ = 600, 900							Second relaxation: $QCap$ = 600, 900						
W	D	V	Q^{EOQ}	Q'	Q''	Q^*	W	D	V	Q^{EOQ}	Q'	Q''	Q^*
2	257.0	4,004.5	99.0	50.2	206.7	50.2	2	254.7	3,910.3	100.4	61.5	210.2	61.5
3	263.6	3,995.6	67.2	30.7	197.0	30.7	3	263.6	3,995.6	67.2	30.7	197.0	30.7
5	215.5	3,510.3	186.0	198.7	268.6	186.0	5	216.2	3,610.1	185.5	191.7	267.3	185.5
8	268.3	4,116.9	169.7	11.1	190.0	11.1	8	268.3	4,116.9	169.7	11.1	190.0	11.1
10	217.5	3,683.5	213.5	131.2	265.2	131.2	10	215.8	3,495.2	214.2	147.2	268.2	147.2
11	204.6	3,163.0	197.4	200.6	285.4	197.4	11	204.6	3,163.0	197.4	200.6	285.4	197.4
13	210.6	3,615.8	217.3	156.0	275.6	156.0	12	223.5	3,936.9	106.0	153.9	255.9	106.0
14	263.9	4,026.6	87.5	28.4	196.5	28.4	13	204.8	3,265.8	219.1	193.8	284.9	193.8
15	200.5	3,211.1	247.9	209.8	291.2	209.8	14	198.0	3,169.2	120.3	269.1	295.0	120.3
16	240.2	4,384.0	74.4	83.1	230.6	74.4	16	231.9	4,049.3	79.5	123.5	243.5	79.5
19	203.2	3,179.8	53.2	313.0	287.4	53.2	19	263.6	4,178.6	13.4	91.2	196.7	13.4
20	209.0	3,576.4	245.2	163.0	278.0	163.0	20	209.0	3,576.4	245.2	163.0	278.0	163.0

The upper bound to errors was 6.4% and 6.5% for first and second Lagrangian relaxation for instances with R = 3, which are higher than instances with R = 1. This might be explained by an increment in duality gaps instead of a heuristic error. For more details of complete results, see Tables 8–13 in Appendix B.

7. Conclusions and Managerial Insights

This research paper is focused on studying a simultaneous model addressing inventory and location decisions, with stochastic demands assuming a periodic review policy and probabilistic constraints of inventory capacity. We determine the location of warehouses from a strategic perspective while taking into consideration several inventory concerns such as costs and constraints. Note that the safety stock and order quantity costs and decisions are also integrated.

The model is built on mathematical expressions for safety stocks, probabilistic inventory capacity constraints, and average cyclic inventory costs. These expressions are considered when on-hand inventory level is lower than reorder inventory level after R periods. This produces an additional ordering quantity called undershoot, which is a relevant issue widely reviewed and researched in the inventory control field.

Furthermore, the fact that the Lagrangian relaxation approach can be applied to solve the ILP with periodic review and stochastic inventory capacity constraints is a significant contribution to the field of study. It is important to mention that a set of inequalities and a local search Lagrangian heuristic are taken into account to provide effective solutions, similar to some related previous studies.

It is shown that upper bounds of the Lagrangian relaxation approach increase when review periods are larger while observing more stable behavior. This is based on a numerical application of small real size instances. This result may be explained because of an increment in duality gaps rather than the heuristic error.

In terms of managerial insights, an integrated inventory location model with a periodic review is suggested to be implemented when the supply chain network topology is analyzed, and the review period of inventory control decisions is significant (2 or more days). Review periods involve a prominent usage of storage space at warehouses or distribution centers, which affects the strategic network topology. Nevertheless, it is not enough to solely analyze benefits related to inventory costs for periodic review versus continuous review policies, or evaluating the variation in the review period of the inventory control policy. As a consequence, strategic network topology costs, such as facility location and transportation costs, must be included in the analysis. Finally, the solution approach and line of research are enhanced, as this allows modeling of several significant issues in inventory control, simultaneously with facility location issues, within the scope of supply chain network design problems.

Further research is suggested in the analysis of different Lagrangian relaxations that may lead to better dual lower bounds and duality gaps. Issues such as different periodic inventory control policies, a multicommodity scenario,

Table 6: Results for first relaxation.

Prob no.	R	$QCap$	FC	FV	DCs opened (∗)	DCs closed (∗)	No. of open DCs	Upper Bound	Lower Bound	% Gap	Lag iter	CPU time (s)
1	1	600	0.7	0.7	10, 11	8, 14	5	2,023,248	2,003,836	0.97	1785	39
4	1	600	0.7	1.0	10, 20	14	6	2,070,318	2,056,800	0.66	1555	34
22	1	600	1.0	0.7	10	14	5	2,161,246	2,131,154	1.41	1011	22
25	1	600	1.0	1.0	2, 3, 5, 8, 14	N/A	5	2,221,538	2,202,020	0.89	1705	37
28	1	600	1.0	1.3	10, 20	14	6	2,282,509	2,263,675	0.83	1343	30
46	1	600	1.3	1.0	None	None	5	2,358,720	2,330,384	1.22	1453	31
49	1	600	1.3	1.3	None	None	5	2,422,577	2,406,503	0.67	1037	23
50	1	900	0.7	0.7	10	14	5	2,022,576	2,002,329	1.01	1906	40
53	1	900	0.7	1.0	10, 20	14	6	2,070,318	2,057,794	0.61	1439	31
71	1	900	1.0	0.7	10	14	5	2,161,246	2,130,454	1.45	1164	24
74	1	900	1.0	1.0	2, 3, 5, 8, 14	N/A	5	2,221,538	2,200,616	0.95	1832	39
77	1	900	1.0	1.3	10, 20	14	6	2,282,509	2,263,675	0.83	1343	30
95	1	900	1.3	1.0	None	None	5	2,358,720	2,330,567	1.21	997	21
98	1	900	1.3	1.3	None	None	5	2,422,577	2,406,503	0.67	1037	23
99	3	600/900	0.7	0.7	12	15,16	11	2,746,539	2,626,621	4.57	1791	51
102	3	600/900	0.7	1.0	None	None	12	2,884,336	2,736,967	5.38	2002	59
120	3	600/900	1.0	0.7	7,12	15,16,19	11	3,063,005	2,910,209	5.25	2707	76
123	3	600/900	1.0	1.0	2,3,5,8,10,11,13, 14,15,16,19,20	N/A	12	3,224,884	3,040,498	6.06	1879	54
126	3	600/900	1.0	1.3	1,12	15	13	3,380,411	3,144,904	7.49	2829	95
144	3	600/900	1.3	1.0	None	None	12	3,565,432	3,340,563	6.73	2372	69
147	3	600/900	1.3	1.3	12	None	13	3,765,373	3,481,191	8.16	1894	57

(∗): with respect to base case, Prob no 25, 74, and 123, respectively; N/A, not applicable.

Table 7: Results for second relaxation.

Prob no.	R	$QCap$	FC	FV	DCs opened (∗)	DCs closed (∗)	No. of open DCs	Upper Bound	Lower Bound	% Gap	Lag iter	CPU time (s)
148	1	600	0.7	0.7	11	8	5	2,023,248	2,004,405	0.94	1330	103
151	1	600	0.7	1.0	20	None	6	2,070,318	2,058,653	0.57	1372	206
169	1	600	1.0	0.7	None	None	5	2,161,246	2,126,896	1.62	1071	83
172	1	600	1.0	1.0	2, 3, 5, 8, 10	N/A	5	2,222,254	2,197,156	1.14	818	65
175	1	600	1.0	1.3	14	10	5	2,285,395	2,264,942	0.90	867	97
193	1	600	1.3	1.0	None	None	5	2,360,924	2,310,342	2.19	773	49
196	1	600	1.3	1.3	14	10	5	2,422,577	2,400,707	0.91	855	54
197	1	900	0.7	0.7	11	8	5	2,023,248	2,002,202	1.05	1378	110
200	1	900	0.7	1.0	14,20	10	6	2,072,384	2,058,187	0.69	1329	188
218	1	900	1.0	0.7	None	None	5	2,161,246	2,126,900	1.61	1318	102
221	1	900	1.0	1.0	2, 3, 5, 8, 10	N/A	5	2,222,254	2,197,156	1.14	818	68
224	1	900	1.0	1.3	14	10	5	2,285,395	2,264,942	0.90	867	102
242	1	900	1.3	1.0	None	None	5	2,360,924	2,310,342	2.19	773	50
245	1	900	1.3	1.3	14	10	5	2,422,577	2,400,707	0.91	855	55
246	3	600/900	0.7	0.7	None	12	11	2,762,632	2,624,245	5.27	813	51
249	3	600/900	0.7	1.0	1	12	12	2,902,314	2,738,089	6.00	1653	102
267	3	600/900	1.0	0.7	7	12,16	11	3,071,115	2,908,436	5.59	1463	94
270	3	600/900	1.0	1.0	2,3,5,8,10,11,12 13,14,16,19,20	N/A	12	3,241,325	3,040,813	6.59	1574	98
273	3	600/900	1.0	1.3	15	None	13	3,396,774	3,160,391	7.48	1428	83
291	3	600/900	1.3	1.0	15	16	12	3,592,914	3,343,519	7.46	1539	91
294	3	600/900	1.3	1.3	1,15	19	13	3,761,791	3,481,305	8.06	1666	101

(∗): with respect to base case, Prob no 172, 221, and 270, respectively; N/A, not applicable.

TABLE 8: Results for the first relaxation, R=1 and $QCap$=600.

Prob no.	FC	FV	DCs opened (∗)	DCs closed (∗)	No. of open DCs	Upper Bound	Lower Bound	% Gap	Lag iter	CPU time (s)
1	0.7	0.7	10, 11	8, 14	5	2,023,248	2,003,836	0.97	1785	39
2	0.7	0.8	10, 20	14	6	2,041,897	2,022,987	0.93	1363	29
3	0.7	0.9	10, 20	14	6	2,056,136	2,041,066	0.74	1611	35
4	0.7	1.0	10, 20	14	6	2,070,318	2,056,800	0.66	1555	34
5	0.7	1.1	10, 20	14	6	2,085,017	2,071,114	0.67	1499	33
6	0.7	1.2	20	None	6	2,099,931	2,089,759	0.49	1397	31
7	0.7	1.3	10, 20	14	6	2,116,294	2,104,572	0.56	1633	36
8	0.8	0.7	10	14	5	2,068,799	2,051,432	0.85	1554	34
9	0.8	0.8	None	None	5	2,092,841	2,071,195	1.05	1827	38
10	0.8	0.9	10	14	5	2,109,062	2,091,839	0.82	2139	46
11	0.8	1.0	10, 20	14	6	2,125,723	2,109,966	0.75	1775	39
12	0.8	1.1	10, 20	14	6	2,140,422	2,115,810	1.16	1415	32
13	0.8	1.2	20	None	6	2,154,840	2,144,656	0.47	2015	44
14	0.8	1.3	10, 20	14	6	2,171,699	2,161,893	0.45	1830	41
15	0.9	0.7	10	14	5	2,115,023	2,091,648	1.12	1554	33
16	0.9	0.8	10	14	5	2,144,384	2,114,680	1.40	1036	22
17	0.9	0.9	10	14	5	2,155,285	2,136,610	0.87	1788	38
18	0.9	1.0	None	None	5	2,175,811	2,153,626	1.03	2038	44
19	0.9	1.1	10, 20	14	6	2,195,827	2,177,076	0.86	1686	37
20	0.9	1.2	20	None	6	2,209,749	2,183,209	1.22	1470	32
21	0.9	1.3	10, 20	14	6	2,227,104	2,212,965	0.64	1688	37
22	1.0	0.7	10	14	5	2,161,246	2,131,154	1.41	1011	22
23	1.0	0.8	None	None	5	2,184,295	2,155,778	1.32	1910	40
24	1.0	0.9	None	None	5	2,202,343	2,177,796	1.13	2718	58
25	1.0	1.0	2, 3, 5, 8, 14	N/A	5	2,221,538	2,202,020	0.89	1705	37
26	1.0	1.1	20	None	6	2,250,748	2,218,640	1.45	1474	32
27	1.0	1.2	20	None	6	2,264,658	2,242,602	0.98	1274	28
28	1.0	1.3	10, 20	14	6	2,282,509	2,263,675	0.83	1343	30
29	1.1	0.7	10	14	5	2,207,470	2,170,288	1.71	929	20
30	1.1	0.8	None	None	5	2,230,023	2,194,687	1.61	1131	24
31	1.1	0.9	10	14	5	2,247,732	2,219,852	1.26	2022	43
32	1.1	1.0	None	None	5	2,267,266	2,244,557	1.01	1834	40
33	1.1	1.1	12	3	5	2,299,007	2,265,427	1.48	1389	31
34	1.1	1.2	12	3	5	2,319,104	2,289,564	1.29	1452	32
35	1.1	1.3	None	None	5	2,331,122	2,307,567	1.02	1804	40
36	1.2	0.7	10	14	5	2,253,693	2,208,455	2.05	1296	28
37	1.2	0.8	None	None	5	2,275,750	2,235,681	1.79	937	20
38	1.2	0.9	None	None	5	2,293,798	2,264,077	1.31	1043	22
39	1.2	1.0	None	None	5	2,312,993	2,288,491	1.07	956	21
40	1.2	1.1	10	14	5	2,346,628	2,311,584	1.52	1531	34
41	1.2	1.2	None	None	5	2,364,918	2,336,833	1.20	1548	34
42	1.2	1.3	None	None	5	2,376,850	2,357,551	0.82	1288	28
43	1.3	0.7	None	None	5	2,299,884	2,247,278	2.34	1011	22
44	1.3	0.8	None	None	5	2,321,478	2,276,995	1.95	1032	22
45	1.3	0.9	None	None	5	2,339,525	2,300,269	1.71	1064	22
46	1.3	1.0	None	None	5	2,358,720	2,330,384	1.22	1453	31
47	1.3	1.1	10	14	5	2,392,852	2,357,628	1.49	1823	40
48	1.3	1.2	12	3	5	2,408,450	2,382,329	1.10	1337	29
49	1.3	1.3	None	None	5	2,422,577	2,406,503	0.67	1037	23

(∗): with respect to base case, Prob no 25; N/A, not applicable.

Table 9: Results for the first relaxation, R=1 and $QCap$=900.

Prob no.	FC	FV	DCs opened (∗)	DCs closed (∗)	No. of open DCs	Upper Bound	Lower Bound	% Gap	Lag iter	CPU time (s)
50	0.7	0.7	10	14	5	2,022,576	2,002,329	1.01	1906	40
51	0.7	0.8	10, 20	14	6	2,041,655	2,014,576	1.34	3631	76
52	0.7	0.9	10, 20	14	6	2,056,095	2,034,976	1.04	955	20
53	0.7	1.0	10, 20	14	6	2,070,318	2,057,794	0.61	1439	31
54	0.7	1.1	10, 20	14	6	2,085,017	2,070,541	0.70	1426	31
55	0.7	1.2	20	None	6	2,099,808	2,085,217	0.70	1454	32
56	0.7	1.3	10, 20	14	6	2,116,294	2,104,572	0.56	1633	36
57	0.8	0.7	10	14	5	2,068,799	2,050,586	0.89	2355	50
58	0.8	0.8	None	None	5	2,092,809	2,071,120	1.05	1913	40
59	0.8	0.9	10	14	5	2,109,062	2,090,298	0.90	2993	63
60	0.8	1.0	10, 20	14	6	2,125,723	2,109,420	0.77	1818	39
61	0.8	1.1	10, 20	14	6	2,140,422	2,127,241	0.62	1157	25
62	0.8	1.2	20	None	6	2,154,717	2,144,883	0.46	1847	40
63	0.8	1.3	10, 20	14	6	2,171,699	2,161,893	0.45	1830	40
64	0.9	0.7	None	None	5	2,116,940	2,091,450	1.22	1980	41
65	0.9	0.8	None	None	5	2,138,537	2,115,098	1.11	1170	25
66	0.9	0.9	10	14	5	2,155,285	2,136,615	0.87	1094	23
67	0.9	1.0	None	None	5	2,175,811	2,155,136	0.96	1277	27
68	0.9	1.1	20	None	6	2,195,646	2,176,603	0.87	1428	31
69	0.9	1.2	20	None	6	2,209,626	2,196,774	0.59	1478	32
70	0.9	1.3	10, 20	14	6	2,227,104	2,212,965	0.64	1688	37
71	1.0	0.7	10	14	5	2,161,246	2,130,454	1.45	1164	24
72	1.0	0.8	None	None	5	2,184,264	2,153,972	1.41	2043	43
73	1.0	0.9	10	14	5	2,201,509	2,179,809	1.00	1256	26
74	1.0	1.0	2, 3, 5, 8, 14	N/A	5	2,221,538	2,200,616	0.95	1832	39
75	1.0	1.1	10, 20	14	6	2,253,266	2,224,064	1.31	1227	27
76	1.0	1.2	20	None	6	2,264,535	2,244,660	0.89	1154	25
77	1.0	1.3	10, 20	14	6	2,282,509	2,263,675	0.83	1343	30
78	1.1	0.7	10	14	5	2,207,470	2,169,785	1.74	965	21
79	1.1	0.8	None	None	5	2,229,992	2,195,949	1.55	948	20
80	1.1	0.9	10	14	5	2,247,732	2,218,807	1.30	992	21
81	1.1	1.0	None	None	5	2,267,266	2,245,445	0.97	1924	42
82	1.1	1.1	10	3	5	2,297,032	2,264,559	1.43	931	21
83	1.1	1.2	12	3	5	2,319,104	2,287,954	1.36	1303	28
84	1.1	1.3	None	None	5	2,331,122	2,307,567	1.02	1804	39
85	1.2	0.7	10	14	5	2,253,693	2,207,597	2.09	979	21
86	1.2	0.8	None	None	5	2,275,719	2,236,741	1.74	989	21
87	1.2	0.9	None	None	5	2,293,798	2,260,875	1.46	1452	31
88	1.2	1.0	None	None	5	2,312,993	2,288,393	1.07	976	21
89	1.2	1.1	10	14	5	2,346,628	2,311,784	1.51	1518	33
90	1.2	1.2	12	3	5	2,363,777	2,336,760	1.16	1514	33
91	1.2	1.3	None	None	5	2,376,850	2,357,551	0.82	1288	28
92	1.3	0.7	None	None	5	2,299,850	2,246,840	2.36	946	20
93	1.3	0.8	None	None	5	2,321,446	2,275,764	2.01	870	18
94	1.3	0.9	None	None	5	2,339,525	2,303,075	1.58	1047	22
95	1.3	1.0	None	None	5	2,358,720	2,330,567	1.21	997	21
96	1.3	1.1	10	14	5	2,392,852	2,351,971	1.74	2396	52
97	1.3	1.2	10	3	5	2,410,607	2,382,015	1.20	1195	26
98	1.3	1.3	None	None	5	2,422,577	2,406,503	0.67	1037	23

(∗): with respect to base case, Prob no 74; N/A, not applicable.

Table 10: Results for the first relaxation, R=3 and $QCap$=600; $QCap$=900.

Prob no.	FC	FV	DCs opened (∗)	DCs closed (∗)	No. of open DCs	Upper Bound	Lower Bound	% Gap	Lag iter	CPU time (s)
99	0.7	0.7	12	15,16	11	2,746,539	2,626,621	4.57	1791	51
100	0.7	0.8	1,12	15,16	12	2,821,579	2,668,238	5.75	1516	43
101	0.7	0.9	12	16	12	2,851,136	2,698,879	5.64	2071	59
102	0.7	1.0	None	None	12	2,884,336	2,736,967	5.38	2002	59
103	0.7	1.1	1,12	11,15	12	2,924,387	2,767,487	5.67	1652	49
104	0.7	1.2	12	None	13	2,988,536	2,804,060	6.58	1764	53
105	0.7	1.3	1,12	15	13	3,005,910	2,839,190	5.87	1904	59
106	0.8	0.7	1,12	15,16,19	11	2,855,967	2,722,479	4.90	1860	53
107	0.8	0.8	None	15	11	2,927,783	2,764,501	5.91	2232	64
108	0.8	0.9	1,12	15,19	12	2,965,225	2,796,807	6.02	1931	55
109	0.8	1.0	None	None	12	2,997,852	2,838,973	5.60	1625	48
110	0.8	1.1	1,12	11,15	12	3,033,260	2,867,744	5.77	2254	67
111	0.8	1.2	12	None	13	3,111,402	2,910,038	6.92	1814	54
112	0.8	1.3	1,12	15	13	3,130,744	2,946,207	6.26	1783	54
113	0.9	0.7	7	15,16	11	2,965,807	2,817,936	5.25	1645	46
114	0.9	0.8	1	11,15	11	3,027,515	2,862,289	5.77	1798	51
115	0.9	0.9	1,12	15,19	12	3,077,708	2,893,453	6.37	1905	54
116	0.9	1.0	None	None	12	3,111,368	2,939,921	5.83	1789	52
117	0.9	1.1	1,12	11,15	12	3,147,729	2,972,010	5.91	1991	59
118	0.9	1.2	12	None	13	3,234,269	3,017,179	7.20	1793	53
119	0.9	1.3	1,12	15	13	3,255,577	3,048,281	6.80	2170	65
120	1.0	0.7	7,12	15,16,19	11	3,063,005	2,910,209	5.25	2707	76
121	1.0	0.8	None	15	11	3,138,137	2,958,396	6.08	1838	52
122	1.0	0.9	1,12	15,19	12	3,190,192	2,992,886	6.59	1964	55
123	1.0	1.0	2,3,5,8,10,11,13, 14,15,16,19,20	N/A	12	3,224,884	3,040,498	6.06	1879	54
124	1.0	1.1	1,12	11,15	12	3,267,793	3,074,058	6.30	1797	53
125	1.0	1.2	12	None	13	3,357,135	3,122,780	7.50	1882	56
126	1.0	1.3	1,12	15	13	3,380,411	3,144,904	7.49	2829	95
127	1.1	0.7	7	15,16	11	3,176,424	3,007,463	5.62	2633	74
128	1.1	0.8	12	15,19	11	3,226,169	3,055,590	5.58	2583	74
129	1.1	0.9	1,12	15,19	12	3,302,676	3,092,201	6.81	1647	47
130	1.1	1.0	None	None	12	3,338,400	3,141,701	6.26	2035	61
131	1.1	1.1	7,12	11,16	12	3,380,893	3,173,092	6.55	2466	81
132	1.1	1.2	1,12	19	13	3,479,214	3,228,727	7.76	1706	59
133	1.1	1.3	12	None	13	3,519,640	3,263,281	7.86	2507	80
134	1.2	0.7	1,12	15,16,19	11	3,265,744	3,098,273	5.41	2484	71
135	1.2	0.8	1,7,12	11,15,16,19	11	3,347,472	3,151,445	6.22	2130	61
136	1.2	0.9	1,12	15,19	12	3,415,160	3,190,416	7.04	2025	58
137	1.2	1.0	None	None	12	3,451,916	3,242,612	6.45	1981	58
138	1.2	1.1	7,12	11,16	12	3,493,527	3,275,587	6.65	2420	71
139	1.2	1.2	1,12	19	13	3,600,037	3,331,035	8.08	2534	76
140	1.2	1.3	1,12	15	13	3,630,078	3,374,142	7.59	2022	61
141	1.3	0.7	1	11,15	11	3,386,827	3,191,709	6.11	3215	91
142	1.3	0.8	1,7,12	11,15,16,19	11	3,449,723	3,243,352	6.36	2728	77
143	1.3	0.9	1,12	15,19	12	3,527,644	3,288,221	7.28	1698	49
144	1.3	1.0	None	None	12	3,565,432	3,340,563	6.73	2372	69
145	1.3	1.1	7,12	11,16	12	3,606,160	3,376,022	6.82	2576	76
146	1.3	1.2	1,12	19	13	3,720,860	3,423,069	8.70	2089	73
147	1.3	1.3	12	None	13	3,765,373	3,481,191	8.16	1894	57

(∗): with respect to base case, Prob no 123; N/A, not applicable.

Table 11: Results for the second relaxation, R=1 and $QCap$=600.

Prob no.	FC	FV	DCs opened (∗)	DCs closed (∗)	No. of open DCs	Upper Bound	Lower Bound	% Gap	Lag iter	CPU time (s)
148	0.7	0.7	11	8	5	2,023,248	2,004,405	0.94	1330	103
149	0.7	0.8	20	None	6	2,041,897	2,024,218	0.87	1344	138
150	0.7	0.9	20	None	6	2,056,136	2,041,853	0.70	999	106
151	0.7	1.0	20	None	6	2,070,318	2,058,653	0.57	1372	206
152	0.7	1.1	14,20	10	6	2,093,276	2,070,168	1.12	3156	375
153	0.7	1.2	12,14,20	3	7	2,113,500	2,086,609	1.29	1680	198
154	0.7	1.3	20	None	6	2,120,860	2,104,350	0.78	1025	124
155	0.8	0.7	None	None	5	2,068,799	2,051,220	0.86	1017	79
156	0.8	0.8	13,14,20	3,5	6	2,096,264	2,072,203	1.16	1186	110
157	0.8	0.9	None	None	5	2,109,062	2,091,920	0.82	859	92
158	0.8	1.0	20	None	6	2,126,971	2,108,184	0.89	1283	145
159	0.8	1.1	14,20	10	6	2,140,929	2,125,481	0.73	895	105
160	0.8	1.2	14,20	10	6	2,154,840	2,144,731	0.47	907	112
161	0.8	1.3	20	None	6	2,171,699	2,162,440	0.43	1071	123
162	0.9	0.7	None	None	5	2,115,023	2,087,493	1.32	1693	133
163	0.9	0.8	14	10	5	2,138,568	2,110,547	1.33	1105	86
164	0.9	0.9	None	None	5	2,155,285	2,136,130	0.90	881	69
165	0.9	1.0	14	10	5	2,175,811	2,155,991	0.92	890	97
166	0.9	1.1	14,20	10	6	2,195,839	2,178,961	0.77	1759	194
167	0.9	1.2	14,20	10	6	2,209,749	2,194,826	0.68	865	93
168	0.9	1.3	20	None	6	2,231,671	2,214,517	0.77	948	100
169	1.0	0.7	None	None	5	2,161,246	2,126,896	1.62	1071	83
170	1.0	0.8	14	10	5	2,184,295	2,151,423	1.53	1092	83
171	1.0	0.9	None	None	5	2,201,509	2,175,341	1.20	984	76
172	1.0	1.0	2, 3, 5, 8, 10	N/A	5	2,222,254	2,197,156	1.14	818	65
173	1.0	1.1	14	10	5	2,257,526	2,219,496	1.71	818	86
174	1.0	1.2	14,20	10	6	2,264,658	2,241,920	1.01	823	91
175	1.0	1.3	14	10	5	2,285,395	2,264,942	0.90	867	97
176	1.1	0.7	None	None	5	2,207,470	2,165,608	1.93	1225	98
177	1.1	0.8	14	10	5	2,230,023	2,191,819	1.74	831	66
178	1.1	0.9	None	None	5	2,247,732	2,214,688	1.49	808	54
179	1.1	1.0	14	10	5	2,267,266	2,244,320	1.02	1060	85
180	1.1	1.1	14	10	5	2,303,254	2,264,255	1.72	806	83
181	1.1	1.2	14	10	5	2,319,191	2,292,148	1.18	877	106
182	1.1	1.3	14	10	5	2,331,122	2,308,181	0.99	925	108
183	1.2	0.7	None	None	5	2,253,693	2,203,834	2.26	1314	88
184	1.2	0.8	None	None	5	2,283,055	2,231,213	2.32	1341	102
185	1.2	0.9	None	None	5	2,293,956	2,252,404	1.84	827	54
186	1.2	1.0	None	None	5	2,314,701	2,283,059	1.39	815	64
187	1.2	1.1	None	None	5	2,346,628	2,307,932	1.68	809	70
188	1.2	1.2	12, 14	3,10	5	2,363,777	2,336,492	1.17	845	90
189	1.2	1.3	14	10	5	2,376,850	2,354,471	0.95	833	113
190	1.3	0.7	14	10	5	2,299,884	2,242,034	2.58	1277	84
191	1.3	0.8	None	None	5	2,329,278	2,270,847	2.57	781	50
192	1.3	0.9	None	None	5	2,340,179	2,298,306	1.82	1211	76
193	1.3	1.0	None	None	5	2,360,924	2,310,342	2.19	773	49
194	1.3	1.1	None	None	5	2,392,852	2,346,581	1.97	797	50
195	1.3	1.2	14	10	5	2,410,646	2,371,311	1.66	839	53
196	1.3	1.3	14	10	5	2,422,577	2,400,707	0.91	855	54

(∗): with respect to base case, Prob no 172; N/A, not applicable.

Table 12: Results for the second relaxation, R=1 and $QCap$=900.

Prob no.	FC	FV	DCs opened (∗)	DCs closed (∗)	No. of open DCs	Upper Bound	Lower Bound	% Gap	Lag iter	CPU time (s)
197	0.7	0.7	11	8	5	2,023,248	2,002,202	1.05	1378	110
198	0.7	0.8	20	None	6	2,041,655	2,024,155	0.86	1588	175
199	0.7	0.9	20	None	6	2,056,095	2,041,989	0.69	1078	122
200	0.7	1.0	14,20	10	6	2,072,384	2,058,187	0.69	1329	188
201	0.7	1.1	20	None	6	2,087,050	2,069,874	0.83	3781	474
202	0.7	1.2	14,20	10	6	2,099,808	2,088,349	0.55	897	106
203	0.7	1.3	20	None	6	2,120,860	2,104,350	0.78	1025	115
204	0.8	0.7	None	None	5	2,068,799	2,050,990	0.87	1017	79
205	0.8	0.8	13,14,20	3,5	6	2,095,860	2,072,179	1.14	1712	173
206	0.8	0.9	14	10	5	2,110,888	2,092,099	0.90	1377	150
207	0.8	1.0	20	None	6	2,126,763	2,110,466	0.77	1465	170
208	0.8	1.1	14,20	10	6	2,140,737	2,126,893	0.65	920	109
209	0.8	1.2	14,20	10	6	2,154,717	2,143,841	0.51	913	123
210	0.8	1.3	20	None	6	2,171,699	2,162,509	0.42	1021	121
211	0.9	0.7	None	None	5	2,115,023	2,087,493	1.32	1693	135
212	0.9	0.8	14	10	5	2,138,537	2,110,820	1.31	886	71
213	0.9	0.9	None	None	5	2,155,285	2,136,130	0.90	881	71
214	0.9	1.0	14	10	5	2,175,811	2,154,123	1.01	806	90
215	0.9	1.1	14,20	10	6	2,195,646	2,179,059	0.76	972	117
216	0.9	1.2	14,20	10	6	2,209,626	2,196,715	0.59	943	110
217	0.9	1.3	20	None	6	2,231,671	2,214,517	0.77	948	106
218	1.0	0.7	None	None	5	2,161,246	2,126,900	1.61	1318	102
219	1.0	0.8	None	None	5	2,190,608	2,153,284	1.73	879	67
220	1.0	0.9	None	None	5	2,201,509	2,175,200	1.21	1139	89
221	1.0	1.0	2, 3, 5, 8, 10	N/A	5	2,222,254	2,197,156	1.14	818	68
222	1.0	1.1	14,20	10	6	2,250,555	2,222,358	1.27	830	93
223	1.0	1.2	14,20	10	6	2,264,535	2,243,913	0.92	882	96
224	1.0	1.3	14	10	5	2,285,395	2,264,942	0.90	867	102
225	1.1	0.7	None	None	5	2,207,470	2,165,608	1.93	1225	96
226	1.1	0.8	14	3	5	2,236,425	2,191,676	2.04	1293	98
227	1.1	0.9	None	None	5	2,247,732	2,214,688	1.49	808	54
228	1.1	1.0	14	10	5	2,267,266	2,241,419	1.15	904	70
229	1.1	1.1	None	None	5	2,300,405	2,259,632	1.80	830	63
230	1.1	1.2	14	10	5	2,319,191	2,283,489	1.56	884	78
231	1.1	1.3	14	10	5	2,331,122	2,308,181	0.99	925	108
232	1.2	0.7	None	None	5	2,253,693	2,203,840	2.26	1004	67
233	1.2	0.8	None	None	5	2,283,055	2,231,213	2.32	1341	104
234	1.2	0.9	None	None	5	2,293,956	2,252,404	1.84	827	53
235	1.2	1.0	None	None	5	2,314,701	2,283,059	1.39	815	63
236	1.2	1.1	None	None	5	2,346,628	2,306,865	1.72	811	65
237	1.2	1.2	14	10	5	2,364,918	2,333,821	1.33	935	100
238	1.2	1.3	14	10	5	2,376,850	2,354,471	0.95	833	115
239	1.3	0.7	14	10	5	2,299,850	2,242,041	2.58	1249	85
240	1.3	0.8	None	None	5	2,329,278	2,270,854	2.57	781	51
241	1.3	0.9	None	None	5	2,340,179	2,298,306	1.82	1211	78
242	1.3	1.0	None	None	5	2,360,924	2,310,342	2.19	773	50
243	1.3	1.1	None	None	5	2,392,852	2,346,581	1.97	797	50
244	1.3	1.2	12	10	5	2,422,495	2,365,429	2.41	786	50
245	1.3	1.3	14	10	5	2,422,577	2,400,707	0.91	855	55

(∗): with respect to base case, Prob no 221; N/A, not applicable.

Table 13: Results for the second relaxation, R=3 and $QCap$=600; $QCap$=900.

Prob no.	FC	FV	DCs opened (∗)	DCs closed (∗)	No. of open DCs	Upper Bound	Lower Bound	% Gap	Lag iter	CPU time (s)
246	0.7	0.7	None	12	11	2,762,632	2,624,245	5.27	813	51
247	0.7	0.8	1	16	12	2,821,579	2,667,098	5.79	1700	102
248	0.7	0.9	1	19	12	2,852,741	2,699,063	5.69	1800	109
249	0.7	1.0	1	12	12	2,902,314	2,738,089	6.00	1653	102
250	0.7	1.1	1	11	12	2,924,387	2,767,645	5.66	1674	101
251	0.7	1.2	15	None	13	2,988,536	2,808,140	6.42	1526	94
252	0.7	1.3	15	None	13	3,028,174	2,839,398	6.65	1534	94
253	0.8	0.7	7	12,16	11	2,860,498	2,719,105	5.20	851	54
254	0.8	0.8	None	19	11	2,919,636	2,764,674	5.61	2577	153
255	0.8	0.9	1	19	12	2,965,225	2,797,354	6.00	1691	101
256	0.8	1.0	1	12	12	3,017,797	2,839,010	6.30	1703	102
257	0.8	1.1	1	11	12	3,038,856	2,869,832	5.89	1917	123
258	0.8	1.2	15	None	13	3,111,402	2,913,305	6.80	1436	86
259	0.8	1.3	7,15	16	13	3,136,899	2,946,396	6.47	1599	95
260	0.9	0.7	None	12	11	2,972,986	2,814,084	5.65	1240	79
261	0.9	0.8	None	19	11	3,021,814	2,862,296	5.57	1713	107
262	0.9	0.9	1	19	12	3,077,708	2,895,100	6.31	2142	134
263	0.9	1.0	1	12	12	3,133,280	2,939,919	6.58	1595	105
264	0.9	1.1	1	11	12	3,153,325	2,972,015	6.10	1771	119
265	0.9	1.2	15	None	13	3,234,269	3,018,470	7.15	1491	97
266	0.9	1.3	15	None	13	3,273,907	3,053,394	7.22	1687	108
267	1.0	0.7	7	12,16	11	3,071,115	2,908,436	5.59	1463	94
268	1.0	0.8	15	10,16	11	3,122,842	2,956,507	5.63	2108	129
269	1.0	0.9	1	19	12	3,190,192	2,993,937	6.56	1776	113
270	1.0	1.0	2,3,5,8,10,11,12 13,14,16,19,20	N/A	12	3,241,325	3,040,813	6.59	1574	98
271	1.0	1.1	1	11	12	3,267,793	3,074,195	6.30	1497	90
272	1.0	1.2	15	None	13	3,357,135	3,123,635	7.48	1793	106
273	1.0	1.3	15	None	13	3,396,774	3,160,391	7.48	1428	83
274	1.1	0.7	7	12,16	11	3,176,424	3,006,561	5.65	1631	102
275	1.1	0.8	None	12	11	3,243,314	3,054,626	6.18	2187	134
276	1.1	0.9	1	19	12	3,302,676	3,092,207	6.81	1806	110
277	1.1	1.0	1	12	12	3,364,246	3,141,710	7.08	1610	97
278	1.1	1.1	15	3	12	3,377,059	3,176,389	6.32	1350	83
279	1.1	1.2	1,15	19	13	3,479,214	3,228,800	7.76	1691	100
280	1.1	1.3	15	None	13	3,519,640	3,267,390	7.72	1667	100
281	1.2	0.7	None	16	11	3,268,978	3,101,566	5.40	1734	107
282	1.2	0.8	None	19	11	3,328,346	3,152,607	5.57	1645	99
283	1.2	0.9	1	19	12	3,415,160	3,190,419	7.04	1772	110
284	1.2	1.0	1	12	12	3,479,729	3,242,618	7.31	1715	107
285	1.2	1.1	15	3	12	3,489,521	3,278,575	6.43	1542	97
286	1.2	1.2	1,15	19	13	3,600,037	3,333,966	7.98	1428	83
287	1.2	1.3	15	None	13	3,642,507	3,374,388	7.95	1491	86
288	1.3	0.7	1	3,19	11	3,389,315	3,194,564	6.10	1805	108
289	1.3	0.8	None	19	11	3,430,524	3,248,659	5.60	1706	101
290	1.3	0.9	1	19	12	3,527,644	3,288,238	7.28	1589	94
291	1.3	1.0	15	16	12	3,592,914	3,343,519	7.46	1539	91
292	1.3	1.1	15	11	12	3,618,080	3,380,594	7.02	1630	97
293	1.3	1.2	1,15	19	13	3,720,860	3,439,130	8.19	1549	88
294	1.3	1.3	1,15	19	13	3,761,791	3,481,305	8.06	1666	101

(∗): with respect to base case, Prob no 270; N/A, not applicable.

Table 14: TC_{ij}, fixed transportation cost between the warehouse i and the customer j.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	68015	26816	58294	85997	91121	23524	78853	27326	94451	93276	71013	8557	32141	18954	54919	42018	68610	27355	46763	49274
2	54038	17036	45491	72276	78255	30363	66441	16354	78457	78339	70782	12676	20651	4233	43819	26047	54243	41907	47661	41910
3	57369	56031	7346	70964	29915	59932	19578	35692	47128	32015	63671	55129	31019	48718	64344	32777	54938	90004	54232	29825
4	82616	99538	53516	87042	26639	107004	37444	83516	43137	18196	96759	102914	78869	94900	97864	74447	79407	137435	96435	72772
5	87898	57221	44189	105728	70870	18453	58614	29490	96491	82442	31581	32253	29929	44092	82409	54293	87054	60700	7508	22878
6	25140	16929	53845	42810	81590	60808	72476	39255	61106	72848	96432	42929	40589	26281	14209	20466	26270	61391	75337	62142
7	34478	15893	40983	52744	70906	49068	60749	25176	60212	65563	82049	32716	26160	16594	28605	9228	34371	58596	61419	47673
8	20228	23619	60514	36564	86669	68864	78273	47567	60883	76029	104668	50753	48786	34271	5997	27425	22206	66376	83672	70092
9	38479	15979	68192	53608	97859	61322	87970	46848	78428	90386	103439	42643	49723	29094	19554	35769	40581	49842	80463	72160
10	80170	83491	29855	90526	3778	80441	9131	60634	54537	25055	66543	79814	55819	75627	90235	60023	77310	115246	67091	44528
11	11595	31295	60191	28373	83808	74935	76416	51706	53628	71239	107874	57344	52189	40622	9348	28742	13708	74858	87939	72342
12	71842	57524	13554	87537	39199	44878	26654	28595	67891	50207	43357	46204	24220	46508	73906	40661	69965	81471	34138	9675
13	97465	76822	39310	113620	53169	46412	43091	45847	92847	70832	17915	57044	43225	64128	97425	64942	95768	88456	23429	21367
14	34706	47575	29307	47199	46518	69820	40347	42049	30947	35054	85268	58677	38954	46038	45728	23021	31973	89510	71647	49201
15	94150	54770	66854	112399	96851	13350	84307	40691	113649	105275	54025	27473	44176	44214	82895	63865	94266	37596	32293	48848
16	29475	22028	39006	47545	67089	54632	57675	29050	53348	60048	84464	39064	28969	23358	27613	5697	28909	65484	65006	48979
17	99639	66306	56804	117649	81916	21246	69979	41154	109403	94751	29856	39191	42234	53504	92775	66308	98969	61074	11134	34885
18	57060	17006	51224	75148	84036	29779	72151	21396	83666	84193	73290	11081	25946	7844	45023	31161	57551	36114	49591	46375
19	56743	32264	27250	74521	60120	31303	47981	3957	69365	63077	57615	23341	1855	20618	53015	23087	55849	58424	37566	24292
20	80126	38060	73004	97611	105687	29464	93292	42082	109198	108456	78131	22652	46859	33345	64848	56693	81098	13325	54296	61885

Table 15: TC_{ij}, fixed transportation cost between the warehouse i and the customer j.

	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	72703	46345	20056	7194	71962	40887	55999	68423	15895	77913	51138	66081	61667	88720	78259	7130	54735	16090	66690	18564
2	59896	32132	4388	22583	65202	41304	42203	61033	19380	66009	50540	50795	47891	77046	62396	23055	44916	23003	50919	34467
3	12655	47280	46809	67126	39077	50892	52574	34509	67591	20727	51493	13120	1080	31167	28561	68615	68854	57878	20505	80578
4	40747	84238	93075	114940	67474	94809	84558	65384	113495	40476	92116	45715	48924	35934	41371	116378	103072	105559	48616	128358
5	54963	67485	43129	37684	38332	1532	77383	36511	54368	56118	11800	61753	50373	64751	77672	40026	84310	23831	66644	48355
6	65579	6844	26423	50436	83459	69100	13254	78692	33844	73573	76975	46932	52373	84079	50683	49982	17137	53494	42477	58912
7	53805	13002	15787	42474	69250	55275	23113	64524	31564	61417	62771	38071	40552	72249	46050	42643	31577	42339	35664	53304
8	71466	11707	34548	57559	91017	77437	10366	86216	39456	79630	85277	51524	58447	89858	52884	56915	9951	61452	46160	65098
9	81025	24610	30347	45995	94838	74107	28431	90274	25146	88676	83259	63908	67756	99492	68444	44760	17046	53525	59964	50673
10	15444	73761	73701	91424	37258	66133	77686	35136	94441	11018	62448	34035	28269	7461	41415	93181	95016	80307	40807	104908
11	69811	13687	40589	64991	92078	81842	4985	87229	47713	78125	89038	48593	57252	87874	47529	64506	14555	67768	42385	73164
12	22497	56518	44668	57148	23049	31712	64333	18238	64121	24664	30897	33291	20010	34716	49202	59096	77560	45148	39923	70447
13	42714	80483	62612	64748	13022	26551	89118	14372	78518	39686	17560	59050	45461	44690	74564	67053	100508	50814	66004	76197
14	34342	31044	44511	70270	62625	66935	32661	57937	63196	42636	70294	11842	23986	51396	14324	71063	50787	65286	4942	82664
15	79666	72250	44455	22254	66323	29406	82329	64441	43747	82236	38126	80939	72192	91657	95662	23949	83066	17824	83921	25769
16	50761	10221	22392	49197	69358	59052	19135	64542	38312	58713	65807	33067	37527	69272	39652	49442	31501	48240	29774	60203
17	66871	78711	52857	40988	45971	13180	88766	45280	60537	67235	16532	74642	63075	75057	90579	43240	94227	28783	79556	48808
18	65653	35278	9184	18268	69654	43302	45070	65635	14186	71638	53009	56610	53741	82651	67937	18313	45447	22026	56548	29203
19	41689	36925	18806	35350	47186	31688	46534	42772	38408	47291	38457	37032	30730	58265	51347	36829	55721	27296	39593	48791
20	87326	59120	34706	11175	83610	49294	68171	80515	21967	92147	59495	81289	76643	102786	93319	9241	63619	24619	81807	3923

demand backorders and backlogging features, and service level optimization may be considered for future investigation. Additionally, we encourage the examination of other Lagrangian heuristics, such as simple versions of Ant Colony Optimization and GRASP, among other well-known heuristics and metaheuristics. In terms of supply chain network design issues, having in mind the present inventory location modeling structure, different distribution strategies may be studied simultaneously with inventory planning aspects (e.g., direct shipments and cross-docking).

Appendix

A. Algorithms

See Algorithms 1 and 2.

Let I be the index set of the N smallest $\overline{\Delta}_i$

Let J be the index set of smallest costs between warehouses i and customer j

x^*: best feasible solution found (primal)

UB, LB: best upper and lower bound found for optimal objective function (primal)

UB^k: upper bound found at each iteration k based on Lagrangian heuristic

LB^k: optimal value or lower bound found at each iteration k for Lagrangian function (21)

$\overline{x}^k$: optimal solution of the relaxed subproblems (SP_i^{1k} and SP_i^{2k}) at each iteration k

$\tilde{x}^k$: feasible heuristic solution found at each iteration k based on Lagrangian heuristic

See Algorithm 3.

B. Results for the First and the Second Relaxation

See Tables 8, 9, 10, 11, 12, and 13.

C. Fixed Transportation Costs

See Tables 14 and 15.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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